Lecture 38: Applications of product theorem

I. A remark on "interpolation"

We have seen how to construct functions on a region $U$ with prescribed zeroes at a set of points in $U$ (that has no limit point in $U$). Can we prescribe arbitrary values? In fact, we can do much better!

**Theorem** Let $U \subset \mathbb{C}$ be open/connected, $A \subset U$ a set with no limit point in $U$, $m: A \to \mathbb{Z}_{\geq 0}$ a function, and $\{w_n, a\} \in \prod_{a \in A} \mathbb{C}^{m(a)+1}$ (i.e., for each $a$, a choice of complex numbers $w_0, a, \ldots, w_{m(a)}, a$). Then there exists an $f \in \text{Hol}(U)$ such that

$$
\frac{f^{(n)}(a)}{n!} = w_{n,a} \quad (\forall a \in A, n \in \{0, m(a)\} \cap \mathbb{Z}).
$$

(i.e., local power series center)
Proof: By the Weierstrass product theorem,

\[(*) \exists g \in \text{Hol}(U) \text{ s.t. } \text{ord}_z(g) = m(z)+1 \quad (\forall z \in A),\]

with no other zeroes. We need to do a local calculation, for which we may take

\[\alpha = 0, \quad m := m(\alpha), \quad \omega_n := \omega_{n,\alpha}.\]

Near 0, by (*)

\[g(z) = \sum_{j \geq 1} b_j \cdot z^{m+j} \quad (b_j \neq 0).\]

If \(P(z) := \frac{c_1}{z} + \ldots + \frac{c_{m+1}}{z^{m+1}},\) then

\[g(z) P(z) = (b_1 + b_2 z + b_3 z^2 + \ldots)(c_1 z^m + \ldots + c_{m+1})\]

we want

\[(\omega_0 + \omega_1 z + \ldots + \omega_m z^m) + \text{higher-order terms}\]

Put

\[c_{m+1} = \frac{\omega_0}{b_1},\]

\[c_m = \frac{(\omega_1 - b_2 c_{m+1})}{b_1},\]

\[c_{m-1} = \frac{(\omega_2 - b_3 c_{m+1} - b_2 c_m)}{b_1},\]

\[\vdots\]

\[\Rightarrow \text{gives desired local form to } f := g \cdot P.\]

New Mittag-Leffler's theorem \(\Rightarrow \text{he Mer}(U) \text{ with principal parts } P_a \text{ constructed in this manner, so that } f = g \cdot h \text{ gives the desired function.} \)
II. Little Picard for functions of finite order

Lemma: Given \( f \in \text{Hol} (\mathbb{C}) \) of finite order \( \lambda_f \neq \mathbb{Z} \), and \( \lambda \in f (\mathbb{C}) \), \( f^{-1} (\lambda) \) is an infinite set.

Proof: We may assume \( \lambda = 0 \). If \( f^{-1} (\lambda) = \{ \lambda_1 , \ldots , \lambda_n \} \) then Hadamard product theorem \( \Rightarrow f (z) = e^{h (z)} \prod_{n=1}^{N} (z - \lambda_n) \) where \( h \) is a polynomial of degree \( d \leq L \lambda_f \).

Now (on the one hand)

\[
\text{ord} (e^h) = \limsup_{R \to \infty} \frac{\log \| e^h \|_{\partial R}}{\log R} = d ;
\]

while (on the other) for \( R \gg 0 \)

\[
\| e^h \|_{\partial R} = \left\| \frac{f}{\prod_{n=1}^{N} (z - \lambda_n)} \right\|_{\partial R} \leq \frac{C e^h e^{R^{\lambda_f + e/2}}}{(R/2)^N} \leq C e^{R^{\lambda_f + e/2}}
\]

and

\[
\| e^h \|_{\partial R} \geq \left\| \frac{f}{(R/2)^N} \right\|_{\partial R} \geq \frac{C e^{R^{\lambda_f - e/2}}}{(R/2)^N} \geq C e^{R^{\lambda_f - e}}
\]

So \( \text{ord} (e^h) = \lambda_f \), and \( \{ \lambda \in \mathbb{Z} \} \Rightarrow \lambda_f \neq \mathbb{Z} \).

This is a contradiction.

\[\uparrow\text{it may be necessary to take} \ R \text{in a particular sequence} \rightarrow \text{ or } (\text{due to} \ h (z) \text{ vs.} \ \lim \text{in formula for order}).\]
Theorem: Let \( f \in H^\infty(\mathbb{C}) \) be nonconstant and of finite order. Then

(i) \( f \) assumes all values in \( \mathbb{C} \) except possibly one

(ii) if \( \lambda_f \neq \mathbb{Z} \), \( f \) assumes each of these an infinite number of times, and in particular has infinitely many zeros.

Proof: (ii) is done (by the Lemma) modulo the observation: if \( f \) omits 0, then \( f = e^h \), \( h \) a polynomial by Hadamard, contradicting nonintegrality of the order \( \lambda_f \).

(i) If \( f(z) = e^{g(z)} \), then \( f - e^h \) is nowhere vanishing, so that \( f(z) - e^h(z) \). Now, \( f(z) - e^h(z) \) omits \( e^{-\lambda_f} - e^h \Rightarrow h \) omits \( \log (e^{-\lambda_f} - e^h) + 2\pi i \mathbb{Z} \Rightarrow h - \beta \) is nowhere vanishing for \( \beta \in \log (e^{-\lambda_f} - e^h) + 2\pi i \mathbb{Z} \). But Hadamard \( \Rightarrow h \) polynomial \( \Rightarrow h - \beta \) polynomial \( \Rightarrow h - \beta \) constant \( \Rightarrow h \) constant \( \Rightarrow f \) constant.
III. Ugly functions on $\mathbb{D}_1$

In Lecture 35, we discussed the existence of functions on any region $V \subseteq \mathbb{C}$ which are nowhere regular on the boundary $\partial V$. This was an application of Weierstrass products; now we will discuss one very concrete way to produce such functions if $V = \mathbb{D}_1$, involving power series (not product theorems).

**Theorem (Hadamard)**

*Given*

- $\{p_n\} \subseteq \mathbb{Z}_+$ increasing sequence with $\frac{P_{n+1}}{P_n} > \lambda$ ($\forall n$) for some $\lambda \in (1, \infty)$
- $\{a_n\} \subseteq \mathbb{C}$ s.t. $f(z) = \sum a_n z^{p_n}$ has radius of convergence 1.

Then no point of $\partial \mathbb{D}_1$ is regular for $f$.

**Proof:** Suppose otherwise; wlog wma $f$ extends to $F \in Hol(\mathbb{D}_1 \cup \overline{\mathbb{D}(1,\epsilon)}) =: \overline{U}$.
Pick \( N \in \mathbb{Z}^+ \) s.t. \( \frac{N+1}{N} < \lambda \), and set

\[
\psi(x) := \frac{x^N + x^{N+1}}{2}.
\]

Note that
- \( \psi(1) = 1 \)
- \( x \in D_1 \setminus \{1\} \Rightarrow |\psi(x)| = \frac{1}{2} |x^N| + |x^{N+1}| < \frac{1}{2} |x^N| : 2 \leq 1 \)

\[
\Rightarrow \psi(D_1) \subset U
\]

\[
\Rightarrow 1 \in \psi(D_{1+\delta}) \subset U
\]

(for some \( \delta > 0 \))

\[
\Rightarrow \psi(D_{1+\delta}) \supset N = \text{small nbhd. of } 1.
\]

Set \( G := \mathcal{F} \circ \psi \in \mathcal{H}(D_{1+\delta}) \), and write

\[
G(x) = \sum_{m=0}^{\infty} Y_m x^m
\]

on \( D_1 \)

\[
f(\psi(x)) = \sum_{n \geq 1} a_n \left( \frac{x^N + x^{N+1}}{2} \right)^p
\]

The \( n \)th term of the sum has powers of \( x \) from \( x^{N_p} \) to \( x^{(N+1)p} \), and the \((n+1)\)st has powers from \( x^{N_p} \) to \( x^{(N+1)p} \). Since

\[
\frac{p+n}{p_n} > \frac{N+1}{N} \quad \text{and} \quad (N+1)p_n < Np_n,
\]

there is no overlapping of powers of \( x \), and the two power series in (ae)
on equal, in particular
\[ \sum_{n=1}^{N} a_n \psi(z)^{p_n} = \left( \sum_{n=0}^{N+1} p_n \right) \psi(z)^{p_N} \]

(assumed)

When convergence \((N \to \infty)\) of the RHS on \(D_{1+\delta} \Rightarrow\)
convergence of the LHS \((N \to \infty)\) on \(D_{1+\delta} \Rightarrow\)
\[ \sum_{n \geq 1} a_n w^n \text{ converges for } w \in \psi(D_{1+\delta}) \Rightarrow \]
\[ \sum_{n \geq 1} a_n w^n \text{ converges for } w \in N \Rightarrow \]

radius of convergence \(> 1\) (a contradiction).

So here's the simplest example of "ugly function":

Example // Let \(F(z) := \sum_{n \geq 1} z^{2^n} \). The radius
of convergence is \((\limsup_{n \to \infty} |1/n|)^{-1} = (\lim_{n \to \infty} 1/n^{1/2})^{-1} = 1\)
\((= \text{ converges absolutely \& uniformly on } D_{1/2}). \text{ Further,}\)
\(p_n = 2^n \text{ satisfies } \frac{p_{n+1}}{p_n} = 2 > \frac{3}{2} =: \lambda. \text{ So the}\)
\(\lambda\text{-Theorem applies, and } D_1 \text{ is a "maximal domain of}\)
\(\psi\text{-holomorphicity" for } F. //\)