Lecture 4: Topology of the Complex Plane

I. Topological spaces

We begin with some generalities.

**Definition** A topological space is a pair $(X, \mathcal{S})$, where $X$ is a set (of "points"), and $\mathcal{S}$ is a collection of subsets of $X$ including:

(a) $X$ itself and the empty set $\emptyset$
(b) finite intersections $\bigcap_{j=1}^{n} U_j$ if each $U_j \in \mathcal{S}$
(c) arbitrary unions $\bigcup_{j \in J} U_j$ if each $U_j \in \mathcal{S}$. $\mathcal{S}$ is called a topology on $X$; members of $\mathcal{S}$ are called open sets (⇔ complement is closed).
Remarks: • notation for complement of \( U \in \Sigma \) is \( U^c \) or \( X \setminus U \)

• we will frequently write "\( X \)" instead of "\((X, \Sigma)\)" for topological spaces.

Some more terminology:

• \( X \) is Hausdorff if

\[ \forall \text{ distinct } x, y \in X, \exists \text{ disjoint open } U, V \in \Sigma \text{ with } \overline{U} \ni x \text{ and } \overline{V} \ni y \]

• A basis (or base) for the topology is a "generating set" \( B \subseteq \Sigma \); i.e. every \( U \in \Sigma \) is a union of elements of \( B \).

• Let \( S \subseteq X \) be a subset (not necessarily belonging to \( \Sigma \)). We denote

\[
\left( S \supseteq \right):= \{ x \in X | \exists U \in \Sigma \text{ s.t. } U \subseteq S \}
\]

\[ b := \{ x \in X | \forall U \in \Sigma, \emptyset \neq \emptyset \neq U \cap S \} \]

\[ \overline{S} := \{ x \in X | \forall U \in \Sigma, U \cap S \neq \emptyset \} \cap D S \]

\[ \text{ acc } (S) := \{ x \in X | \forall U \in \Sigma, \forall n \geq 0 U^{(n)} \cap S \neq \emptyset \} \]
• \( S \) is connected \( \iff \) \( S \) cannot be written as the disjoint union \( U \sqcup V \) of nonempty \( U, V \in \mathcal{E} \).

Remark: Any \( S \) has a unique decomposition into connected components (\( \mathcal{E} := \) maximal connected subsets = clopen sets).

• \( S \) is compact \( \iff \) every open cover has a finite subcover: i.e.,

\[
\begin{align*}
S \subseteq \bigcup_{j \in J} U_j & \quad (\text{each } U_j \in \mathcal{E}) \\
\Downarrow \\
\exists \{J_0, J_1, \ldots, J_n\} \subseteq J \quad s.t. \quad S \subseteq \bigcup_{i=1}^{n} U_{j_i}
\end{align*}
\]

Distance.

Given a distance function

\[ d: \mathbb{X} \times \mathbb{X} \to \mathbb{R}, \]

it makes sense to take as basis for \( \mathcal{E} \) the "open disks"

\[ D(x_0, r) := \{ x \in \mathbb{X} \mid d(x, x_0) < r \}. \]

If \( \mathcal{E} \) is constructed in this way, the triple \((\mathbb{X}, \mathcal{E}, d)\) is called a metric space.

**Properties:**

- symmetric
- nonnegative
- \( d(x, y) = 0 \iff x = y \)
- \( A \) inequality
Remark: I will also use the notation
\[ D(x_0, r) := \{ x \in X \mid d(x, x_0) \leq r \} \]
and \[ D^*(x_0, r) := \{ x \in X \mid 0 < d(x, x_0) < r \} \].

### Examples

<table>
<thead>
<tr>
<th>( X )</th>
<th>name of metric</th>
<th>( d(z_1, z_2) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbb{C} )</td>
<td>Euclidean</td>
<td>(</td>
</tr>
<tr>
<td>( \mathbb{C} )</td>
<td>stereographic</td>
<td>( \frac{2</td>
</tr>
<tr>
<td>( \mathbb{H} )</td>
<td>Poincaré</td>
<td>( \log\left(\frac{</td>
</tr>
<tr>
<td>( \Delta )</td>
<td>Poincaré</td>
<td>( \tanh^{-1}\left(\frac{</td>
</tr>
</tbody>
</table>
II. The complex plane

Henceforth we work on $X = \mathbb{C}$ with the Euclidean metric $d(x,y) = |x - y|$. In particular, open sets contain disks around each of their points.

Examples of subsets of $\mathbb{C}$

1. \[ \mathbb{C}, \quad \mathbb{C} = \text{entire circle} \]
   \[ \mathbb{C} = \text{shaded area} \]
   \[ \text{acc} (\mathbb{C}) = \mathbb{C} = \mathbb{C} \cap \mathbb{C} \]

2. \[ \mathbb{C}, \quad \mathbb{C} \]

Here $\mathbb{C} = \overline{\mathbb{C}}$ (is closed)
and $\text{acc}(\mathbb{C}) = \emptyset$
3 (upper half plane) \[ \mathcal{S} = \mathbb{H} \]

\[ \mathcal{S} = \mathbb{D} \quad \text{(open set)} \]

4 \[ \mathcal{S} \]

\[ \text{acc } (\mathcal{S}) = \{ \alpha \}, \quad \text{whether or not } \alpha \in \mathcal{S} \]

Remarks:

(i) The definition of \( \alpha \in \text{acc} (\mathcal{S}) \) says that for any \( \varepsilon > 0 \), \( D^*(\alpha, \varepsilon) \cap \mathcal{S} \) is nonempty. In effect, this means that \( D(\alpha, \varepsilon) \cap \mathcal{S} \) contains infinitely many points. For \( \mathcal{S} = \mathbb{C} \), \( \mathcal{S} \cap \text{acc} (\mathcal{S}) \subset \mathcal{S} \) but we need not have \( \text{acc} (\mathcal{S}) \subset \mathcal{S} \) or \( \mathcal{S} \leq \text{acc} (\mathcal{S}) \) (cf. Examples 2 & 4 above).
(ii) Given \( S \subseteq C \), the open subsets of \( S \) are the \( \{ S \cap U \} \) for \( U \in \Sigma \) (i.e. \( U \) open in \( C \)); the closed subsets are their complements (in \( S \)). But \( S \cap U \) (resp. \( S \setminus S \cap U \)) may not be open (resp. closed) in \( C \). In this sense, openness and closedness are relative properties. (From the form of the definition of compactness, it is clear that this is an absolute property.)

Examples:

<table>
<thead>
<tr>
<th>( R = S )</th>
<th>( C = X )</th>
<th>( h = S )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(open in ( R ) (but not in ( C ))</td>
<td>(closed in ( h )) (but not in ( C ))</td>
<td>(but not in ( C ))</td>
</tr>
</tbody>
</table>
III. Limits

Given $f : \mathbb{R} \to \mathbb{C}$, $x \in \mathbb{R}$ and $\beta \in \mathbb{C}$, define

$$\lim_{x \to a} f(x) = \beta \iff \forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } f(x \cap D(a, \delta)) \subseteq D(\beta, \varepsilon).$$

Now suppose $x \in \mathbb{R}$; then

$f$ is continuous at $x \iff \lim_{x \to a} f(x) = f(x).$

A sequence $\{x_n\}$ amounts to a (continuous) function

$$f : \left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\} \to \mathbb{C},$$

and the limit of the sequence (if it exists) is given by

$$\lim_{n \to \infty} x_n = w \iff \lim_{x \to 0} f(x) = w.$$
The accumulation points (or limit points) of the sequence are all the limits of subsequences \( \{x_k\} \). (Note that the accumulation points \( \text{acc}(\{x_n\}) \) of the set \( \{x_n\} \) may be different – e.g., for constant sequences, empty.)

A sequence is Cauchy \( \iff \) \( \forall \epsilon > 0, \exists N \in \mathbb{Z}_+ \) s.t. \( n, m \geq N \Rightarrow |x_n - x_m| < \epsilon \).

\[ \Rightarrow \] the sequence converges.

(\text{since } \mathbb{C} \text{ is complete})

IV. Remarks on connectedness

Connectedness was defined in III.

Real numbers

\( I \subset \mathbb{R} \) is connected \( \iff \) \( I \) is an interval.

Consequently, any bounded-above (resp. below), nonempty subset \( S \subset \mathbb{R} \) has a least upper bound (resp. greatest lower bound), since the set

\( \dagger \) See Ahlfors
of all upper bounds for $S$ is easily shown to be connected and closed. More precisely:

$\inf (S) := \sigma \in \mathbb{R}$ s.t. (i) $\sigma \geq$ every element of $S$

(always belongs to $S$) (ii) if $\tau \geq$ every element of $S$

then $\tau \geq \sigma$ as well.

**Complex numbers**

For $S \subseteq \mathbb{C}$, a path in $S$ is a continuous function $f : [0, 1] \to \mathbb{C}$, and the important thing to remember is:

**Proposition** Assume $S \subseteq \mathbb{C}$ open. Then

$S$ is connected $\iff$ $S$ is pathwise connected.

(i.e. $\forall \alpha, \beta \in S$ $\exists$ path with $f(0) = \alpha$, $f(1) = \beta$)

**Proof (idea):**

$->$ $S_0 \subseteq S$ maximal pathwise connected

$\Rightarrow S_0$ open & closed (use open ball $B$)

$\Rightarrow S_0$ maximal connected (so $S_0 = S$).

$<->$ If $S = U \cup V$ and $f(0) = \alpha$, $f(1) = \beta$,

we get $f^{-1}(U) \cup f^{-1}(V) = [0, 1]$, a contradiction.

$\Box$
A connected open set in \( \mathbb{C} \) is called a region, and it is on such sets that we will generally want to study holomorphic functions.

**V. Boundedness**

We say that

\[
\mathcal{S} \subseteq \mathbb{C} \text{ is bounded } \iff |s| \leq C \in \mathbb{R}_+ \quad (\forall s \in \mathcal{S}).
\]

**Theorem (Bolzano-Weierstrass)**

\( \mathcal{S} \subseteq \mathbb{C} \) bounded infinite \( \implies \) \( \text{acc}(\mathcal{S}) \neq \emptyset \).

**Proof:** Let \( \{r_n\} \subseteq \mathbb{R} \) be a bounded sequence. We can extract either a nonincreasing or non-decreasing subsequence: if the subset of \( \{r_n\} \) consisting of “elements \( \geq \) all subsequent terms” is infinite, it gives a nonincreasing sequence.
Otherwise, one can construct a non-decreasing one. Now take g.l.b or l.u.b of that subsequence; this exists and must be its limit $s$.

Next let $\{s_n\} \subseteq \mathbb{R}$ be an arbitrary sequence of distinct elements (which exists as $|\mathbb{R}| = \infty$). We may extract a subsequence with convergent real part, then a sub-subsequence with convergent imaginary part (by the result for $\{u_n\}$ above). The latter $\{s_{n_k}\}$ must then not only converge to some $s \in \mathbb{C}$, but have $s \in \text{acc}(\{s_{n_k}\}) \leq \text{acc}(\mathbb{R})$ since all the $s_{n_k}$ are distinct.

**Theorem**

Let $K \subseteq \mathbb{C}$ be a subset.

The following are equivalent:

(i) $K$ is compact

(ii) Any $\{z_n\} \subseteq K$ has a point of accumulation in $K$ (equivalently: a convergent subsequence with limits in $K$)

(iii) $K$ is closed and bounded
Proof: $(ii) \Rightarrow (i)$: HW

$(iii) \Rightarrow (ii)$: see proof of Bolzano-Weierstrass

$(i) \Rightarrow (iii)$:
- **Boundedness**: take $U_n = \{ x \mid 1 < n \}$.
  - $\cup U_n = C \supset K$ ($\forall U_n$ gives open cover).
- **Compactness**: $K \subset U_n$ for some $N$.
- **Closedness**: Assume $K$ not closed.
  - Take $p \in \partial K$, $\notin K$, and $U_n = \overline{B}(p, \frac{1}{n})^c$.
  - Then $\cup U_n = C \backslash \{ p \} \supset K$.
- $K$ compact $\Rightarrow K \subset U_n$ which is false as $U_n^c$ must (as $p \in K$) contain a point of $K$.

Remark: Any accumulation point of $\{ x_n \} \subset K$ compact is contained in $K$, since $K$ is closed.

**Theorem**: Let $K \subset \mathbb{C}$ be compact, with nonempty closed nested subsets $(K_n)_{n \geq 1}$ $\supseteq \cdots$. Then $\cap K_n \neq \emptyset$.

**Proof**: The $K_n$ are compact (since closed, $\delta$ inherit boundedness from $K$).
- For each $m$, choose $x_m \in K_m$; $K$ contains an accumulation point of $\{ x_m \}$, say $\delta$.
- This $\delta$ is an accumulation point of tails $x_m$, $m \geq n$ $\subseteq K_n$, hence $\cap K_n \ni \delta$ (4th). Hence $\cap K_n$ contains $\delta$. 

APPENDIX: An interesting non-Hausdorff space

A more general definition of the limit of a sequence, for arbitrary \((X, \Sigma)\), is

\[
\lim_{n \to \infty} x_n = x \iff \forall U \in \Sigma, \exists N \text{ such that } n \geq N \implies x_n \in U
\]

HW: Uniqueness of \( \lim_{n \to \infty} x_n \) is implied by Hausdorffness of \((X, \Sigma)\).

(Hence, we can show that a topological space is non-Hausdorff by exhibiting a "non-unique limit point".)

Example

Writing

\[\Delta = D(0,1), \quad \Delta^\times = D^\times(0,1), \quad \tau(s) = \frac{\log(s)}{2\pi},\]

define a lattice \( \Gamma_s \subseteq \mathbb{C}^2 \) for each \( s \in \Delta \) by

- \( s \in \Delta^\times \): \( \Gamma_s := \mathbb{Z} \langle (i, \tau(s)), (0), (1), (0) \rangle \)
- \( s = 0 \): \( \Gamma_0 := \mathbb{Z} \langle (0), (0) \rangle \).

Then \( \bigcup_{s \in \Delta} \Gamma_s \times \{s\} =: \Gamma \subseteq \mathbb{C}^2 \times \Delta \)

is reasonably nice (e.g. union of smooth submanifolds),
but
\[ \overline{X} := \frac{C^2 \times \Delta}{\Gamma} \]
use quotient topology
is non-Hausdorff

Proof (sketch): Let \( a \in \mathbb{Z}\setminus\{0\} \), \( b \in \mathbb{C} \), and
note that for \( s_n := e^{2\pi i (b + ni)/a} \),
\[ \lim_{n \to \infty} s_n = 0. \]
Consider
\[ v_n := a \left( \frac{1}{\lambda s_n} \right) - n \left( \frac{i}{b} \right) = \left( \frac{a}{b} \right). \]
\( \in \Gamma_{s_n} \)
Then
\[ (v_n, s_n) \equiv (0, s_n) \mod \Gamma_{s_n} \]
\[ \downarrow \]
\[ (\in C^2 \times \Delta) \]
\[ \downarrow \]
\[ ((a), s_n) \equiv ((b), s_n) \mod \Gamma_{s_0} \]
\[ \left\langle \text{limits distinct in } \mathbb{R} \right\rangle !!!! \]