Lecture 8: Analytic functions

I. Definition & basic properties

Let $U \subseteq \mathbb{C}$ be open, $f \in \mathcal{O}(U)$.

- $f$ is analytic at $z_0 \in U \iff$
  \[ \exists \{a_n\} \subseteq \mathbb{C} \text{ and } r \in \mathbb{R}_{>0} \text{ s.t.} \]
  \[ \sum_{n=0}^{\infty} a_n (z - z_0)^n \text{ converges absolutely to } f(z) \]
  \[ \text{for all } z \in \mathcal{D}(z_0, r). \]

That is, "$f$ is described by a power series in a neighborhood of $z_0$," or "has a power series expansion at $z_0".

- $f$ is analytic on $U \iff$

  $f$ is analytic at every $z_0 \in U$.

We can also define analyticity on an arbitrary
set $\Delta \subset \mathbb{C}$:

- a given $f \in \mathcal{O}(\Delta)$ is "analytic on $\Delta$" $\iff$

  $\exists \; U \subset \mathbb{C}$ open containing $\Delta$ and $f \in \mathcal{O}(U)$ s.t. $f = \frac{F}{g}$

As you would expect, $\mathcal{O}(U)$ is a $\mathbb{C}$-algebra:

- $f, g \in \mathcal{O}(U) \implies f + g, fg, cf \in \mathcal{O}(U)$
  
  $\quad \forall c \in \mathbb{C}$

  $\quad f/g \in \mathcal{O}(U \setminus \{x : g(x) = 0\})$

**Proof:** Given $z_0 \in U$, $f$ & $g$ are represented by

\[ f(z) = \sum_{r=0}^{\infty} a_r (z - z_0)^r \]

\[ g(z) = \sum_{r=0}^{\infty} b_r (z - z_0)^r \]

The **formal** sum/product/quotient $\left(\text{provided } g(z_0) \neq 0 \iff \text{ord } (g(z)) = 0\right)$ converges on the same/poss. smaller disk, $\forall$ $f/g$ resp.

$fg$ resp. $f/g$ (cf. Lecture 6). Do at each $z_0$.

- $f \in \mathcal{O}(U), g \in \mathcal{O}(V)$ with $g(V) \subset U$

  $\implies f \circ g \in \mathcal{O}(V)$.

**Proof:** Similar argument, using "(B)" at end of lecture 6. \[ \square \]
II. Relation to power series

- \( f(t) \in \mathbb{R}, \Rightarrow f(z) \in \mathbb{C} \),

(There is a corresponding statement for origin replaced by any \( z_0 \).)

**Proof:** Goal: Show \( f \) analytic at \( z_0 \in \mathbb{D}_r \).

Take \( s < r - 1201 \). For \( z \in \mathbb{D}(z_0, s) \),

\[
  f(z) \xrightarrow{\text{by assumption}} \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} 2^{-n-k} (z - z_0)^k
\]

converges absolutely since

\[
  \sum_{n=0}^{\infty} |a_n| \sum_{k=0}^{n} \binom{n}{k} 120^{-n-k} |z - z_0|^k \leq \sum_{n=0}^{\infty} |a_n| (120 + s)^n < r
\]

converges (by hypothesis). Switching order of summation is therefore permissible, and yields

\[
  f(z) = \sum_{k=0}^{\infty} \left( \sum_{n=k}^{\infty} \binom{n}{k} a_n 2^{-n-k} \right) (z - z_0)^k
\]
So we now have
\[ n_r : \mathbb{R}^r \to \text{An}(D_r). \]

III. Zeros of analytic functions

- Assume \( f, g \in \text{An}(U) \) with \( U \) connected.
  Then \( \text{acc}\{z \in U \mid f(z) = g(z)\} \neq \emptyset \Rightarrow f = g \) on \( U \).

**Proof:** From the last proof, if the (unique) power series
representing a function \( F \) at \( z_0 \) has radius of convergence \( r(z_0) < \infty \), then the radius \( r(z) \) for \( z \), nearby cannot be very different:

\[ r(z) \]

power series for \( z \) must converge
on the dotted disk.

If it converges on a much larger disk,
then the one at \( z_0 \) does too for the
same reason.

\[ \Rightarrow \text{"radius of convergence function" } r_f : U \to \mathbb{R}_{>0} \]

either continuous or as everywhere.

(Picture shows \( |r_f(z_0) - r_f(z_1)| \\leq |z_0 - z_1| \).)
Let \( t_0 \in \text{acc} \{ t \in U \mid F(t) = g(t) \} \), write \( F = f - g \), and let \( t \in U \). We want to show \( F(t) = 0 \).

Let \( Y \) be a path (since \( U \) is connected) from \( t_0 \) to \( t \), which for simplicity (and without loss of generality) we can assume linear:

\[
\begin{array}{cccccc}
0 & 1 & 2 & 3 & 4 & 5 \\
\end{array}
\]

(See the beginning of Lecture 5.) Let \( \nu < \frac{1}{2} \) and take \( t_1, t_2, \ldots \) at least within \( \nu \) of each other. The neighborhoods of convergence about each \( t_i \); therefore, contain \( t_{i+1}, t_{i-1} \).

Recall that in the situation where \( F \) is represented by power series at \( t_0 \) with \( t_0 \in \text{acc} \{ t \mid F(t) = 0 \} \), that
The power series is \( z \leq 0 \) (see Lecture 6); this implies
\[
F = 0 \quad \text{on} \quad D(0, b).
\]
Since \( z \leq D(0, b) \), \( z \in \text{acc} \{ F = 0 \} \) implies
\[
F = 0 \quad \text{on} \quad D(z, b).
\]
This contains \( z \); continuing the argument we find
\[
F(z) = 0, \quad \text{done}.
\]

**Corollary:** \( f \in \text{An}(U) \) and \( K \subset U \) compact
\[
\implies f \text{ has only finitely many zeros on } K.
\]

**IV. Relation with complex differentiability**

- \( f \in \text{An}(U) \) with \( |f| \) constant, \( f' = 0 \), \( \text{Re}(f) \) constant or \( \text{Im}(f) \) constant \( \implies f \) constant.

**Proof:** In a moment we will show that analytic \( \implies \) holomorphic, for which we have the statement in the first two cases. If
\[
\text{Re}(f) \text{ is constant, then}
\]
\[
f' = \frac{df}{dz} = \frac{df}{dx} - i \frac{df}{dy} = i \frac{dv}{dx} + \frac{dv}{dy} = -i \frac{dv}{dx} + \frac{dv}{dy} = 0.
\]
\( f \) holomorphic \( \quad u \) constant \( \quad \text{C-R equations} \)
Analytic functions are holomorphic (in fact, infinitely differentiable).

I want to put this in a broader context involving formal differentiation

\[
D : \mathcal{C}[T] \to \mathcal{C}[T]
\]

\[
\sum_{n=0}^{\infty} a_n T^n \mapsto \sum_{n=0}^{\infty} n a_n T^{n-1}
\]

**Theorem**

Let \( f(T) \in \mathcal{C}[T] \), with

\[
a_n(f(T)) = : f^{(n)}(T) \in \text{An}(D^n),
\]

where \( r \) is the radius of convergence of \( f \) at \( T = 0 \).

(a) \( Q \) is closed under \( D \), in fact, \( Df \) has radius of convergence \( r \).

(b) \( \frac{df}{dz}(T) \) is defined and equals \( (DF)(T) \) exists and equals

(c) \( \text{An}(V) \subseteq \text{Hol}(U) \), for any region \( U \).

**Proof:**

(a) \( \limsup |a_n|^{1/n} = \lim_n (\sup |a_n|)^{1/n} = r \).

(i) take log & apply L'Hôpital, OR (to avoid such things)
(ii) write $n = \left( 1 + \left( \frac{1}{n^2} - 1 \right)^n \right) > 1 + \binom{n}{2} \left( \frac{1}{n^2} - 1 \right)^2$

\[
\sqrt{n} > \frac{n(n-1)}{2} \left( \frac{1}{n^2} - 1 \right)^2
\]

\[
0 < \sqrt{n} > \frac{n(n-1)}{2} \left( \frac{1}{n^2} - 1 \right)^2 > 0.
\]

(b) Recall a complex differentiable at $z$ with derivative $a$

\[
\lim_{h \to 0} \left| \frac{f(z+h) - f(z)}{h} - a \right| = 0.
\]

Take $\delta < r - |z|$ and $|h| < \delta$:

then $f(z+h) - f(z) = \sum a_n (z+h)^n - z^n$

\[
= \left( \sum n a_n z^{n-1} \right) h + (\sum a_n f_n (z,h) \langle h, 2n \rangle)
\]

where $|P_n (z,h)| = \left| \sum \binom{n}{k} a_k h^{k-2} \right|$

\[
\leq \sum \binom{n}{k} |a_k| \left| h^{k-2} \right|^{n-1} ^{n-1}
\]

\[
\leq n(n-1) \sum \binom{n-2}{k} \delta^k |a|^{n-2} - 2
\]

\[
= n(n-1) (|a| + \delta) - 2
\]

\[
= n(n-1) r_0 ^{n-2}
\]

Hence, $\left| \frac{f(z+h) - f(z) - \sum a_n z^{n-1}}{h} \right| = \left| h \sum a_n P_n (z,h) \right|$

\[
(f)(z)
\]
\[ |h| \sum |a_n| r_0^{n-2} \xrightarrow{r_0 < r} 0, \]

\text{known convergent as } r_0 < r \text{ and } (n(n-1))^{1/n} \to 1 \text{ done.}

(c) In a neighborhood of every \( z \in U \),
\( f \) is represented by a power series and
(by (b)) \( f' \) exists and is represented by \( q_c((Df)(z)) \).

\[ \boxed{\text{Corollary}} \]
(a) Let \( f \in \text{An}(U) \); then
\( f \) is infinitely complex-differentiable on \( U \).

(b) Let \( f \in \text{An}(D_r) \) be given as \( q_r(f(T)) \).
Then (i) \( f^{(k)}(z) = \gamma (D^k f(T)) = k! a_k + h_k(z) \)
so \( f^{(k)}(0) = k! a_k \), with \( h_k(0) = 0 \).

(ii) \( f \) has a primitive on \( D_r \) (i.e., a function \( F \) satisfying \( F' = f \)),
\[ F = \sum_{n=0}^{\infty} \frac{e_n}{n+1} z^{n+1} \text{ (constant) } \in \text{An}(D_r). \]
Pf. of (6): About each point \( f \) is represented by power series; and we just showed \( f' \) is represented by the formal derivative (in some neighborhood), hence is analytic itself (and we may repeat this process).

IV. Teuber's theorem

We conclude by revisiting the topic of the last lecture, where we had shown that

\[
(*) \quad \text{OS} \Rightarrow \text{CS} \Rightarrow \text{AS}
\]

Ordinary summability \( \Rightarrow \) Cesàro summability \( \Rightarrow \) Abel summability

The problem with this story is that in daily mathematics (not just analysis!), we encounter situations where we need to be able to deduce information (asymptotic behavior, ordinary summability).
about the \( \{a_k\} \) for \( f(0) = \sum a_k 2^k \). Frequently \( f \) may be a function one understands well, for which we decide to compute the \( f(a)/b! \). What can we say about these numbers?

The theory that provides this kind of information is Tauberian theory, and its first result is:

**Theorem (Tauber, 1897)**: \( a_n + n a_n \to 0 \Rightarrow 0 \)

That is, we can (conditionally) go "backwards" in (1)!

**Proof:** \( 0 < |x| < 1 \Rightarrow \)

\[
\left| \sum_{n=0}^{N} a_n - f(x) \right| = \left| \sum_{n=1}^{N} a_n (1-x^n) - \sum_{n=2N+1}^{\infty} a_n x^n \right|
\]

\[
\leq \sum_{n=1}^{N} |a_n (1-x)| + \frac{1}{N} \sum_{n=2N+1}^{\infty} |a_n x^n|
\]

\[
\leq (1-x) \sum_{n=1}^{N} |a_n| + \frac{1}{N} \sup_{n > N} |a_n|
\]

Therefore (taking \( x = 1 - \frac{1}{N} \))
\[
\left| s_N - f(1 - \frac{1}{N}) \right| \leq \frac{1}{N} \sum_{n=1}^{N} \left| a_n \right| + \sup_{n \geq N} \left| a_n \right|
\]

Now \( a_n \to 0 \Rightarrow \text{this} \to 0 \quad (\text{case}) \quad (n \to \infty) \quad (N \to \infty)

While \( A \subseteq \lim_{N \to \infty} f(1 - \frac{1}{N}) \quad (= A) \) exists.

To complete our discussion of these results, we contrast what Abel & Tauber say for a function
\[
f(x) := \sum a_n x^n \quad \text{on} \quad (-1, 1).
\]

\underline{Abel}: \( \sum a_n \) converges \( \Rightarrow f \) has continuous extension
to \( (-1, 1] \) (by setting \( f(1) := \sum a_n \)).

\underline{Tauber}: \( f \) has continuous extension \( \Rightarrow \sum a_n \) converges
to \( (-1, 1] \) (and \( a_n \to 0 \)) \( \to f(1) \).

So Tauber's theorem is a conditional converse.

There's a stronger version due to Littlewood (1911),
relaxing "\( a_n \to 0 \)" to \( |a_n| \leq B \).
Remark: $a_n = \frac{1}{n \log(n)}$ is an example of a sequence satisfying $n a_n \to 0$ but not ordinary summable:
\[ \sum_{n=2}^{\infty} \frac{1}{n \log n} = \infty, \quad \text{essentially by the integral test} \]
\[ \int_{2}^{\infty} \frac{dx}{x \log x} = \int_{\log 2}^{\infty} \frac{du}{u} = \infty. \]

So we really do need both hypotheses in Tauber's theorem to conclude OS.