Lecture 9: Continuation and multivaluedness

I. Exp and log

Recall that we have holomorphic functions

\[
\exp(x + iy) := e^x \cos(y) + i e^x \sin(y) \in \text{Hol}(\mathbb{C})
\]

\[
\log(z) := \log |r| + i \arg(r) \in \text{Hol}(\mathbb{C} \setminus \mathbb{R}_{\leq 0}),
\]

which was proved by checking the C-R equations.

Now write

\[
\overline{\exp(x + iy)} = \exp(-x + iy) = \eta(\exp(T))
\]

\[
\overline{\log(z)} = \log(1/z) = \eta(\log(1/T))
\]

we also had

\[
\cos(T) = \eta \left( \sum_{k=0}^{\infty} \frac{(-1)^k T^{2k}}{(2k)!} \right), \quad \sin(T) = \eta \left( \sum_{k=0}^{\infty} \frac{(-1)^k T^{2k+1}}{(2k+1)!} \right).
\]
The formal relation
\[ \exp(iT) = C(T) + iS(T) \]
\[ \Rightarrow \exp(iy) = \cos(y) + isin(y) = \exp(iy), \]
By calculus, we also have
\[ \exp(x) = \exp(x). \]
Now in \( C(T) \)
\[ \exp((Y_1 + Y_2)T) = \sum_{n=0}^{\infty} \frac{(Y_1T + Y_2T)^n}{n!} \]
\[ = \sum_{n=0}^{\infty} \frac{Y_1T}{k!} \cdot \frac{Y_2T}{(n-k)!} \]
\[ = \left( \sum_{k=0}^{\infty} \frac{Y_1T}{k!} \right) \cdot \left( \sum_{k=0}^{\infty} \frac{Y_2T}{k!} \right) \]
\[ = \exp(Y_1T) \exp(Y_2T), \]
So applying \( z \)
\[ \Rightarrow \exp((Y_1 + Y_2)z) = \exp(Y_1z) \exp(Y_2z) \]
\[ \Rightarrow \exp(Y_1 + Y_2) = \exp(Y_1) \exp(Y_2) \]
\[ \Rightarrow \exp(1) = \exp(x) \exp(iy) = \exp(x) \exp(iy) = \exp(z), \]
\[ \exp(t) \in \mathcal{O}(t). \]

To do the same for \( \log \), by Calculus we know that
\[ \tilde{\log} (1 - z) = \log (1 - z) \quad \text{for} \quad z = x + i \in (-1, 1) \quad \text{real}. \]
Furthermore, \( \exp (\log (1-x)) = 1 - x \) there, and so the formal power series
\[ (*) \quad (\exp \circ f)(T) = 1 + T \]
is sent by \( \eta \) to a function on \( D_1 \) which is identically zero on \( (-1, 1) \). Since \( f \) is even \( (-1, 1) \), by Lecture 6 \( (*) \) is the zero power series, and so
\[ \exp (\tilde{\log} (1 - z)) = 1 - z \quad \text{on} \quad |z| < 1. \]
We can repeat this for \( \tilde{\log} \circ \exp \) on some sufficiently small disc, and so
\[
\log(z) = \log(1 + \exp(i \arg z)) \quad \exp(\log|z|)
\]
\[
= \log(\exp(\log|z| + i \arg z)) \\
= \log|z| + i \arg z \\
= \log(z),
\]
on some neighborhood of \(z = 1\) (and hence on all of \(\mathbb{D}(1,1)\)).

We can do the same thing for \(\log(z)\) on any disk in \(\mathbb{C} \setminus \{0\}\); if the center is \(z_0\), write
\[
\frac{1}{z} = \frac{\frac{1}{z_0}}{1 + \frac{z - z_0}{z_0}} = \frac{1}{z_0} \left(1 + \frac{1}{z_0} (z - z_0) + \frac{1}{z_0^2} (z - z_0)^2 + \cdots\right),
\]
integrate, and add \(\log(z_0)\) as constant. If \(z_0 \in \mathbb{D}(1,1)\), for example, we have that the resulting \(\tilde{\log}(z)\) agrees with \(\log(z)\) at \(z_0\) and has the same derivative \(\frac{1}{z}\) hence by the above
properties of analytic functions must agree on $D(1,1) \cap D(z_0, 1 \leq 1)$. In this way one shows that

$$\log(z) \in \text{An}(\mathbb{C} \setminus \mathbb{R}_{\leq 0}).$$

Finally, from

$$\exp (\log \alpha - \log \beta) - \log \alpha - \log \beta) = \frac{\exp (\log \alpha) - \exp (\log \beta)}{(\exp (\log \alpha) - \exp (\log \beta)) \alpha \beta} = 1,$$

we have

$$\log (\alpha \beta) \equiv \log \alpha + \log \beta \pmod{2\pi i \mathbb{Z}}.$$

Out of this, we can get even more analytic functions:

**Example** By composing analytic functions,

$$\exp(x \log \beta) \in \text{An}(\mathbb{C} \setminus \mathbb{R}_{\leq 0}).$$

Here, $\mathbb{R}_{\leq 0}$ can be replaced by any slit going from $0$ to $\infty$.

Alternatively, instead of deleting the slit,
you can think of $e^x$ as being ambiguous by 
\[ \{ \exp(2\pi i n R) \} , \text{ which consists of finitely many values if and only if } \lambda \in \mathbb{Q} . \]

\[ \text{Example:} \]

Since $\cos(x)$ and $\sin(x)$ belong to $\mathbb{R} \cup \mathbb{C}^1$ by construction, any identity which is true on the level of power series holds also for the functions.

In particular,

\[ \cos(x) = \frac{1}{2} (\exp(ix) + \exp(-ix)) \]

\[ \Rightarrow 2w = e^i + e^{-i} \]

\[ \Rightarrow 0 = e^i - 2w e^i + 1 \]

\[ \Rightarrow e = \frac{2w \pm \sqrt{4w^2 - 4}}{2} = w \pm \sqrt{w^2 - 1} \]

\[ \Rightarrow \arccos(w) = \pi = -i \log(e) = -i \log(w \pm \sqrt{w^2 - 1}) . \]

Now, $w \pm \sqrt{w^2 - 1}$ is an analytic function on $\mathbb{C}$ (choose one: we'll do $\cdot$)
Because \( w^{1/2} \) is of the form in the last example, and \( w^2 - 1 \) maps this region into \( \mathbb{C} \setminus \{ R \geq 0 \} \). Furthermore, considering the equation

\[
\begin{align*}
    w + \sqrt{w^2 - 1} &= r \in \mathbb{R}_{>0} \\
    \sqrt{w^2 - 1} &= r - w \\
    w^2 - 1 &= r^2 - 2rw + w^2 \\
    w &= \frac{r^2 + 1}{2r} = \frac{1}{2} \{ \frac{1}{1 + r} \} \in \mathbb{R}_{\geq 1}
\end{align*}
\]

we see that \( w + \sqrt{w^2 - 1} \) maps the region \( \{ R \leq -1 \} \) into \( \mathbb{C} \setminus \{ R \geq 0 \} \), where we can also define \( \log(w) \) as an analytic function; it follows that the composition is analytic:

\[
\arccos(w) \in \text{An}(\mathbb{C} \setminus \{ R \leq -1 \} \cup \{ R \geq 1 \}).
\]

Remark: We have to do this all by hand, in the absence of something like the inverse mapping theorem (which wouldn't give us analyticity of \( \arccos \) on more than a disk anyway!).
II. Riemann surfaces

Let's have another look at the example — taking (say) $\alpha = \frac{1}{3}$. If we decided to keep pasting power-series together along overlap of neighborhoods, we would quickly get “multivalued” behavior in the functions constructed from log. So the question arises as to what the “existence domain” of a multivalued function over an open set in $\mathbb{C}$ (i.e. the minimal “cover” of $\mathbb{C}$ making the function single-valued) looks like. The resulting “complex 1-manifolds” are called Riemann surfaces.

Example/ Riemann surface of $g(z) = z^{\frac{1}{3}}$ over $D_1$. 
This is some object fitting (as \( \{ z = w^3 \} \)) into the following picture:

\[
\begin{align*}
\begin{array}{c}
\text{\(e^{\frac{2\pi i}{3}}\)} \\
\text{\(e^{\frac{4\pi i}{3}}\)} \\
\text{\(e^{\frac{6\pi i}{3}}\)} \\
\end{array}
\end{align*}
\]

\(z\text{-disk}\)

\(w\text{-disk}\)

\(\{ z = w^3 \}\)

\(\pi\)

\(\pi\)

To construct it, think about following \(z^{\frac{1}{3}}\) around the disk once counterclockwise: when you reach your starting point, the function has become \(2\pi i/3\) times the branch of \(z^{\frac{1}{3}}\) you started with; going around once more, you get \(e^{2\pi i/3} z^{\frac{1}{3}}\); and one more time gets you back to your original branch.

So taking 3 unit disks, slitting them along the positive reals, and gluing them as indicated
we get the "cyclic parking lot"

An easier way to visualize this RS is: it's just the w-disk. The difficulty is in seeing the w-disk "over" the t-disk.
Example 1: Next we construct an existence domain for
\[ \Omega(t) = \sqrt{(t-a)(t-b)(t-c)} = \sqrt{f(t)} \]
on \hat{C} (or \( C \setminus \{a,b,c\} \) if you prefer).

In a neighborhood of \( t_0 = a, b, c \), this looks like the "RS of \( (t-t_0)^2 \) over a disk", which is the same as the construction we just did except with 2 disks replacing 3. Indeed, going once around \( t = a, b, \) or \( c \) takes
\[ f \mapsto -f. \]

Furthermore, because the degree of the polynomial is odd, going once around \( x \) does the same thing. Since

\[ \begin{array}{c}
\text{and}
\end{array} \]

are equivalent, going around 2 points at once
gives no change. So taking 2 Ĉ's and cutting & pasting them as indicated, we end up with a donut-shaped surface on which it becomes well-defined:

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+√f  "open" the cuts and join  "−√f"
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\begin{tikzpicture}
  \node at (0,0) {\rotatebox{90}{$\alpha$}};
  \node at (1,0) {\rotatebox{90}{$\beta$}};
  \node at (2,0) {\rotatebox{90}{$\gamma$}};
  \node at (3,0) {\rotatebox{90}{$\delta$}};
  \node at (4,0) {\rotatebox{90}{$\epsilon$}};
  \node at (5,0) {\rotatebox{90}{$\zeta$}};
  \node at (6,0) {\rotatebox{90}{$\eta$}};
  \node at (7,0) {\rotatebox{90}{$\theta$}};
  \node at (8,0) {\rotatebox{90}{$\iota$}};
  \node at (9,0) {\rotatebox{90}{$\kappa$}};
  \node at (10,0) {\rotatebox{90}{$\lambda$}};
  \node at (11,0) {\rotatebox{90}{$\mu$}};
  \node at (12,0) {\rotatebox{90}{$\nu$}};
  \node at (13,0) {\rotatebox{90}{$\xi$}};
  \node at (14,0) {\rotatebox{90}{$\omicron$}};
  \node at (15,0) {\rotatebox{90}{$\pi$}};
  \node at (16,0) {\rotatebox{90}{$\rho$}};
  \node at (17,0) {\rotatebox{90}{$\sigma$}};
  \node at (18,0) {\rotatebox{90}{$\tau$}};
  \node at (19,0) {\rotatebox{90}{$\upsilon$}};
  \node at (20,0) {\rotatebox{90}{$\phi$}};
  \node at (21,0) {\rotatebox{90}{$\chi$}};
  \node at (22,0) {\rotatebox{90}{$\psi$}};
  \node at (23,0) {\rotatebox{90}{$\omega$}};
  \node at (24,0) {\rotatebox{90}{$\alpha'$}};
  \node at (25,0) {\rotatebox{90}{$\beta'$}};
  \node at (26,0) {\rotatebox{90}{$\gamma'$}};
  \node at (27,0) {\rotatebox{90}{$\delta'$}};
  \node at (28,0) {\rotatebox{90}{$\epsilon'$}};
  \node at (29,0) {\rotatebox{90}{$\zeta'$}};
  \node at (30,0) {\rotatebox{90}{$\eta'$}};
  \node at (31,0) {\rotatebox{90}{$\theta'$}};
  \node at (32,0) {\rotatebox{90}{$\iota'$}};
  \node at (33,0) {\rotatebox{90}{$\kappa'$}};
  \node at (34,0) {\rotatebox{90}{$\lambda'$}};
  \node at (35,0) {\rotatebox{90}{$\mu'$}};
  \node at (36,0) {\rotatebox{90}{$\nu'$}};
  \node at (37,0) {\rotatebox{90}{$\xi'$}};
  \node at (38,0) {\rotatebox{90}{$\omicron'$}};
  \node at (39,0) {\rotatebox{90}{$\pi'$}};
  \node at (40,0) {\rotatebox{90}{$\rho'$}};
  \node at (41,0) {\rotatebox{90}{$\sigma'$}};
  \node at (42,0) {\rotatebox{90}{$\tau'$}};
  \node at (43,0) {\rotatebox{90}{$\upsilon'$}};
  \node at (44,0) {\rotatebox{90}{$\phi'$}};
  \node at (45,0) {\rotatebox{90}{$\chi'$}};
  \node at (46,0) {\rotatebox{90}{$\psi'$}};
  \node at (47,0) {\rotatebox{90}{$\omega'$}};
  \node at (48,0) {\rotatebox{90}{$\omega''$}};
  \node at (49,0) {\rotatebox{90}{$\phi''$}};
\end{tikzpicture}
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Remark: Ahlfors also does RSs of

- $\log(z)$ (in level parking lot)
- $\arccos(z)$ (more complicated; please read it)