We now turn to the part of this course which is closely related to analytic of algebraic number theory, beginning with a "differentiation lemma" whose proof will use three results you may already know (but are described in the Appendix).

Lemma: Given \( I \subset \mathbb{R} \) interior (possibly infinite), \( U \subset \mathbb{C} \) open, and \( f \in C^0_c(I \times U) \) such that

1. \( \int_I f(t, z) \, dt \) is uniformly convergent on any compact \( K \subset U \)
2. \( f \) is analytic in \( z \) (i.e. \( f(z, \cdot) \in \mathcal{Hol}(U) \) for each fixed \( t \in I \)),

we have

\[
F(z) := \int_I f(t, z) \, dt \in \mathcal{Hol}(U)
\]

and

\[
F'(z) = \int_I \partial_t f(t, z) \, dt \quad \text{(where \( \partial_t f \) satisfies the same hypotheses as \( f \) itself)}.
\]

Proof:

Consider \( I_1 \subset I_2 \subset \ldots \subset I \) (\( U_{I_n} = I \)),

\[
\overline{D} = \overline{D}(z_0, R) \subset U.
\]

Cauchy + (ii) \( \Rightarrow f(t, z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z, s)}{s - z} \, ds \quad \text{for } z \in D.\]
\[ F(z) = \frac{1}{2\pi i} \int_{\partial D(0, R_2)} \frac{f(t, z)}{s - z} \, ds \, dt. \]

For \( z \in \overline{D(0_2, R_2)} \), boundness of \( |\frac{f(t, z)}{s - z}| \) on \( I_n \times \partial D \) let us write

\[ F_n(z) = \int_{I_n} f(t, z) \, dt = \frac{1}{2\pi i} \int_{\partial D} \frac{f(t, z)}{s - z} \, ds \, dt = \frac{1}{2\pi i} \int_{\partial D} \frac{1}{s - z} \int_{I_n} f(t, s) \, dt \, ds. \]

This is holomorphic (on \( \overline{D(0, R_2)} \)) by (A) (in the Appendix). By (i), \( F_n(z) = \int_{I_n} f(t, z) \, dt \) is uniformly on \( \overline{D(0, R_2)} \) and hence \( F_n \to F \) normally on \( U \), hence by (C) \( F_n' \to F' \) normally. Finally, by (B) we have \( F_n'(z) = \int_{I_n} \partial_z f(t, z) \, dt \), whose limit is by definition \( \int_{I_n} \partial_z f(t, z) \, dt \).

\[ \text{Proposition: } \Gamma(z) = \int_0^{2\pi} e^{-it} \, dt, \quad z \in \text{hol}(U), \quad U = \{ z \mid \text{Re}(z) > 0 \}. \]

\[ \text{Proof: } \text{Clearly } f(t, z) = e^{-it} \text{ is analytic in } z. \text{ We need to check uniform convergence, which we'll actually do on } K_{a,b} = \{ z \mid 0 < \text{Re}(z) < b \} \text{ when } 0 < a < b < \infty. \text{ In fact, this is clear from finiteness of } \int_0^{2\pi} e^{-it} \, dt \text{ and } \]

\[ \text{\dag We know } D_z f(t, z) \text{ is continuous (for applying B) because we can apply A to the Cauchy integrals } f(t, z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(s, z)}{s - z} \, ds \text{ to get } D_z f(t, z) = \frac{1}{2\pi i} \int_{\partial D} \frac{\partial_z f(s, z)}{s - z} \, ds, \text{ which is evidently continuous in } t \text{ and } z \text{ because the integrand is } \frac{1}{(s - z)^2}. \]
\[ \int_0^\infty e^{-t}t^{b-1}dt \] for any \( a > 0 \) and \( b < \infty \). We also see that \( \Gamma'(z) = \int_0^\infty e^{-t} (\log t) t^{z-1}dt \).

Now, we would like to have an "analytic continuation" of \( \Gamma \) to \( \mathbb{C} \) if possible; and you'll recall that the standard way to build entire functions (\( \Gamma \) won't be one, but \( \frac{\Gamma}{z} \) will) was Weierstrass products?

Begin by integrating by parts:

\[ \Gamma(z+1) = \frac{1}{z} \int_0^\infty e^{-t}t^{z}dt \]

\[ (a = e^{-t}, \quad dv = t^{z-1}dt) \]

\[ (du = e^{t}dt, \quad v = t^{z+1}/(z+1)) \]

\[ = \frac{1}{z} \int_0^\infty e^{-t}t^{z+1}dt \]

\[ = \ldots \]

\[ = \frac{1}{z(z+1) \ldots (z+n)} \int_0^\infty e^{-t}t^{z+n}dt \]

\[ = \frac{\Gamma(z+n+1)}{z(z+1) \ldots (z+n)} \]

\[ \Rightarrow z \Gamma(z) = \Gamma(z+1), \]

which together with \( \Gamma(1) = \int_0^\infty e^{-t}dt = 1 \) gives \( \Gamma(n) = (n-1)! \)

\[ \text{The integral here is holomorphic in } z \text{ for } \Re(z) > -n-1 \text{ (?!)} \]

So this gives an analytic continuation arbitrarily far to the left, but what about an expression for all of \( \mathbb{C} \)?
For $x \in \mathbb{R}^+$, write

\[
\int_0^1 (1 - \frac{t}{x})^n t^{x-1} \, dt = \frac{n^x}{n!} \int_0^1 (1 - y)^n y^{x-1} \, dy
\]

\[
\int_0^1 (1 - y)^n y^{x-1} \, dy = \frac{n^x n!}{x(x+1) \cdots (x+n-1)} \int_0^1 y^{x+n-1} \, dy = \frac{n^x n!}{x(x+1) \cdots (x+n)} = \frac{1}{g_n(x)}.
\]

Now for $t \in [0, n]$,\n
\[
(1 - \frac{t^2}{n^2})^n = 1 - n\frac{t^2}{n^2} + (\frac{t}{n})^4 - (\frac{t}{n})^6 t \cdots \geq 1 - \frac{t^2}{n}
\]

\[
e^t (1 - \frac{t^2}{n^2})^n \geq (1 + \frac{t}{n})^n (1 - \frac{t}{n})^n \geq 1 - \frac{t^2}{n}
\]

\[
e^{\frac{t^2}{n}} \geq 1 + \frac{t^2}{n}
\]

\[
\Rightarrow \quad \frac{t^2}{n} e^{-t} \geq e^{-t} - (1 - \frac{t^2}{n})^n \geq 0. \quad (**)
\]

Since $\int_0^\infty e^{-t} t^{x-1} \, dt < \infty$ (as $\int_0^n \frac{t^2}{n} e^{-t} t^{x-1} \, dt \to 0$) and $\int_0^\infty e^{-t} t^{x-1} \, dt < \infty$ using (**), we can take the limit of (*) to obtain
\[
\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt = \lim_{n \to \infty} \int_0^n e^{-t} t^{x-1} dt \\
= \lim_{n \to \infty} \frac{n!}{n^x (x+1) \cdots (x+n)} \\
= \lim_{n \to \infty} \left\{ n \left( \prod_{k=1}^n \left( 1 + \frac{x}{k} \right) e^{-\frac{\alpha}{k}} \right) e^{x \left( \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n \right)} \right\}^{-1} \\
= \left\{ n \left( \prod_{k=1}^n \left( 1 + \frac{x}{k} \right) e^{-\frac{\alpha}{k}} \right) e^{x \gamma} \right\}^{-1}
\]

where

\[ Y := \lim_{n \to \infty} \left( 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n \right) \]

is Euler's constant. ↑

**Theorem** \( \Gamma(z)^{-1} = e^Y z \prod_{k=2}^\infty \left( 1 + \frac{z}{k} \right) e^{-\frac{z}{k}} \) gives a

meromorphic continuation of \( \Gamma \) to all of \( \mathbb{C} \); moreover, \( \Gamma(z) \) has simple poles at \( z \leq 0 \) (and no zeros at all).

↑ The point is that the LHS converges by the Lemma and the RHS Wurster product is an odd-1 canonical product (so convergent) ⇒ \( Y < \infty \). More directly, \( Y = 1 + \sum_{m \geq 2} \frac{1}{m} + \log(m) - \log(m) \)

with term \( \frac{1}{m} + \log(1 - \frac{1}{m}) = \frac{1}{m} - \frac{1}{m} + \frac{1}{2m^2} - \cdots < \frac{1}{2m^2} \) so converges!
Proof: We proved this works for $x \in \mathbb{R}_+$, and the RHS is well-defined as analytic (converged product). But two analytic functions with the same values on a segment, agree everywhere.

**Corollary 1** \[ \Gamma(x) \Gamma(-x) = \frac{\pi}{\sin(\pi x)} \]

**Proof:** Recall \( \sin(\pi x) = \pi x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2}\right) \); so

\[
\Gamma(x)\Gamma(-x) = e^{-x^2}x^{-1/2-1} \prod_{n=1}^{\infty} \left(1 + \frac{x^2}{n^2}\right)^{-1} \left(1 - \frac{x^2}{n^2}\right)^{-1} e^{\frac{\pi i}{2}} e^{-\frac{\pi i}{2}} \\
= -\frac{1}{x^2} \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2}\right)^{-1} = -\frac{\pi}{x \sin(\pi x)}
\]

and \( \Gamma(-z) = -z \Gamma(-z) \).

Note in particular that this gives

\[ \Gamma(\frac{1}{2}) \Gamma(\frac{1}{4}) = \frac{\pi}{\sin(\pi/2)} = \pi \]

\[ \Rightarrow \quad \Gamma(\frac{1}{2}) = \sqrt{\pi} \]

**Corollary 2** \[ \text{Res}_{-n} \Gamma(z) = \frac{(-1)^n}{n!} \quad \text{for } n \in \mathbb{Z}_{\leq 0} \]

**Proof:** By Corollary 1,
\[
(2\pi)^{(z+n)} \frac{\prod (2\pi+n)}{(\sin \pi z)^n \Gamma (1-z)} = \frac{(-1)^n \pi (2\pi+n)}{(\sin \pi (2\pi+n)) \Gamma (1-z)}
\]

and since (looking at the Weierstrass product)

\Gamma's poles are simple,

\[
\text{Res}_{z=n} (\Gamma(z)) = \lim_{z \to 2\pi} (z+n) \Gamma(z) = \frac{\lim_{z \to 2\pi} \frac{\prod (2\pi+n)}{\sin \pi (2\pi+n)}}{\Gamma (1+n)}
\]

\[
= \frac{(-1)^n}{n!}
\]

---

**Corollary 3**

\[
\prod_{j=0}^{N-1} \Gamma (2 + iN) = \frac{(2\pi)^{\frac{N-1}{2}} \Gamma (N+2)}{N^{\frac{N+2}{2}}}\]

(Gauss's multiplication formula)

---

**Proof:**

LHS & RHS have no zeroes, and simple poles at 0, \(-\frac{1}{N}, -\frac{2}{N}, \ldots\), so the quotient is "exp" of something. Taking into account that the reciprocals of both sides are entire functions of order 1, we find that the quotient is actually of the form \(A_0 \cdot B_0^z\). (!!!)
That is, \( h(z) = A B^z = \frac{\prod_{j=0}^{N-1} \Gamma(z + j/N)}{\Gamma(Nz)} \)

\[ \Rightarrow (B = \frac{h(z+1)}{h(z)}) = \frac{\prod_{j=0}^{N-1} (z + j/N)}{\Gamma(Nz + N)/\Gamma(Nz)} = \frac{\prod_{j=0}^{N-1} (z + j/N)}{\Gamma(Nz + N)}/\Gamma(Nz) = \frac{\prod_{j=0}^{N-1} (z + j/N)}{\Gamma(Nz + N)} \]

\[ = N^{-N}. \]

Now \( A = h(0) = \left( \frac{\prod_{j=0}^{N-1} \Gamma(z)}{\Gamma(Nz)} \right) = N \prod_{j=0}^{N-1} \Gamma(z) > 0, \]

essentially \( \frac{1}{1/2} \)

and \( \left( \frac{A}{N} \right)^2 = \prod_{j=1}^{N-1} \Gamma(\frac{j}{N}) \Gamma(\frac{1-j}{N}) = \frac{\prod_{j=1}^{N-1} \Gamma(\frac{j}{N})}{\prod_{j=1}^{N-1} \Gamma(1-j/N)} = \frac{\prod_{j=1}^{N-1} \Gamma(\frac{j}{N})}{\prod_{j=1}^{N-1} \sin \left( \frac{\pi j}{N} \right)} \]

\[ = \frac{(2\pi)^{N-1}}{\prod_{j=1}^{N-1} (e^{\pi i/N} - e^{-\pi i/N})} = \frac{(2\pi)^{N-1}}{\prod_{j=1}^{N-1} (1 - e^{2\pi i j/N})} \]

\[ \prod e^{\pi i j/N} \prod (1 - e^{2\pi i j/N}) \]

\[ \Rightarrow A = \frac{(2\pi)^{N-1}}{\sqrt{N}} N = N^{1/2} (2\pi)^{N-1/2}. \]
Special case: $N = 2 \Rightarrow$ Legendre duplication formula:

$$\Gamma(x + \frac{1}{2}) \Gamma(x) = \sqrt{2\pi} \frac{\Gamma(2x)}{2^{2x} - 2}$$

$$\Rightarrow \Gamma(x + \frac{1}{2}) = \frac{\Gamma(2x)}{\Gamma(x)} 2^{1-2x} \sqrt{\pi}.$$

So for example (taking $z = -1$)

$$\Gamma(-\frac{1}{2}) = \lim_{z \to -1} \frac{\Gamma(2z)}{\Gamma(z)} 2^{3} \sqrt{\pi} = \frac{1}{2} \frac{\text{Res}_{-2} \Gamma}{\text{Res}_{-1} \Gamma} 8 \sqrt{\pi}$$

$$\Rightarrow \frac{(-1)^{2}/2!}{(-1)!} 4 \sqrt{\pi} = -2 \sqrt{\pi},$$

**Cor. 2**

and this clearly checks with our earlier computation because

$$\Gamma(-\frac{1}{2}) = -\frac{1}{2} \Gamma(-\frac{1}{2}) = -\frac{3}{4}(-2\sqrt{\pi}) = \sqrt{\pi}.$$
Appendix: Three basic results.

(A) \( f(z) := \int_{\gamma} \frac{g(w)}{w-z} \, dw \) is holomorphic on \( U \setminus \gamma \), with derivative \( f'(z) = \int_{\gamma} \frac{g(w)}{(w-z)^2} \, dw \).

(B) Suppose \( F(t, z) \in C^0([a,b] \times U) \), with \( \partial_z F(t, z) \) also defined \& continuous. Then \( G(t) := \int_a^b F(t, z) \, dt \) is differentiable, and \( G'(t) = \int_a^b \partial_z F(t, z) \, dt \).

(C) Let \( F_n \in \text{hol}(U) \) be a sequence converging normally (i.e. uniformly on compact subsets \( K \subset U \)) to \( F \in \text{hol}(U) \). Then \( F_n' \to F' \) (normally).

[Shruth: use Cauchy together with (A).]
Proof of (A): [Note: we don't have to prove $f =$ anything, hence don't have to use Cauchy's formula.]

Pick any $z \in U \setminus Y$, and set $r := d(z_0, \partial U \cup Y)$. For all $w \in \Omega$ and $z \in D(z_0, r)$, $|z-z_0| < r \leq |w-z_0|$, 

$$
|z-z_0| < r \leq |w-z_0| \\
\Rightarrow \left| \frac{z-z_0}{w-z_0} \right| < 1.
$$

Thus,

$$
f(z) := \int_Y \frac{g(w)}{w-z} \\
= \int_Y \left( \sum_{k=0}^{\infty} \frac{(z-z_0)^k g(w)}{(w-z_0)^{k+1}} \right) \, dw \\
= \sum_{k=0}^{\infty} \int_Y \frac{(z-z_0)^k g(w)}{(w-z_0)^{k+1}} \, dw \\
= \sum_{k=0}^{\infty} \left( \int_Y \frac{g(w)}{(w-z_0)^{k+1}} \, dw \right) (z-z_0)^k \\
= \sum_{k=0}^{\infty} a_k (z-z_0)^k.
$$

Hence, $f(z)$ is analytic at $z_0$,

with $f^{(m)}(z_0) = m! a_m$

$$
= m! \int_Y \frac{g(w)}{(w-z_0)^{k+1}} \, dw.
$$