Lecture 17: More on zeta functions

I. Functional equation

The functional equation for \( \zeta(s) \) allows us to understand its behavior on \( \text{Re}(s) < 0 \) in terms of that on \( \text{Re}(s) > 1 \), leaving only the critical strip \( \text{Re}(s) \in [0,1] \) as more mysterious.

For \( m \in \mathbb{Z}_{\geq 2} \), consider the contour

\[
\begin{align*}
- \int_{D_m} G_m(w) \frac{(-w)^s}{w} \, dw &= -2\pi i \sum_{\text{Res, } w = 2\pi i k} \frac{(w^{-1})^{s-1} e^{-2\pi i k}}{1 - e^{-w}} \\
&= -2\pi i \sum_{k=1}^{\infty} e^{-2\pi i k} (2\pi |k|)^{s-1} e^{-\frac{2\pi i k}{|k|}} \frac{1}{1 - e^{-2\pi i k}} \\
&= (2\pi)^s \sum_{\ell=1}^{\infty} \frac{e^{-2\pi i \ell} e^{-\pi^2/4} - e^{2\pi i \ell} e^{-\pi^2/4}}{\ell^{1-s}}.
\end{align*}
\]
Taking the limit as $m \to \infty$, the integral over the inner square goes to $0$, and so for $x \in [0,17]$ we get
\[
H_x(s) = (2\pi)^s \sum_{n=1}^{\infty} \frac{-2i\sin(2\pi n x + \frac{n\pi}{2})}{n^{1-s}}.
\]

For Riemann theta ($x=1$) this gives
\[
-\frac{2\pi i \theta_3(0)}{\Gamma(1-s)} = H_1(s) = -(2\pi)^s \sum_{n=1}^{\infty} \frac{e^{\frac{in\pi}{2}} - e^{-\frac{in\pi}{2}}}{n^{1-s}}
\]
\[
= -2i(2\pi)^s \sin(\frac{\pi s}{2}) \zeta(1-s)
\]

\[\Rightarrow \quad \text{(for } \text{Re}(s) < 0)\]

(11) \quad \zeta(s) = \zeta(1-s) \times \Gamma(1-s) \frac{\sin(\frac{\pi s}{2})}{\pi} (2\pi)^s.

But since we know (by uniqueness of $H_1(s)$) that $\zeta$ is "entire meromorphic", it follows that (11) holds on $\mathbb{C}$.

\[\text{Theorem 2} \quad \zeta(s) = s(s-1)\pi^{-\frac{3}{2}} \Gamma(\frac{3}{2}) \zeta(s).
\]

Then $\zeta$ is an entire function satisfying the functional equation
\[\zeta(s) = \zeta(1-s).\]
Proof: Recall that
\[ \frac{\Gamma'(u)}{\Gamma(u+\frac{1}{2})} = \left( \frac{2^{1-u}}{\pi^{u}} \right) \frac{\Gamma(2u)}{\Gamma(u)} \]
and
\[ \frac{\Gamma'(u)}{\Gamma(u)} = \frac{\pi}{\sin(\pi u)} \cdot \frac{1}{u} \]
Together with (11) these imply that $\zeta(s)$ is invariant under $s \rightarrow 1-s$. \[ \text{[Exercise]} \]
Furthermore, recalling $H_I(s) = -2i \left( \sin(\pi s) \frac{\Gamma(s)\zeta(s)}{\zeta(1-s)} \right)$, we see that $\zeta(s)$ can have poles only at $\mathbb{Z}_{\geq 0}$. But $\sum \frac{1}{n^s}$ is holomorphic for $\text{Re}(s) > 1$, so the only possible pole is at $s = 1$. From (11), we see this is a simple pole, so that $(s-1)\zeta(s)$ is actually entire. Since $\zeta$ is therefore holomorphic on $\text{Re}(s) > 0$, it must (by the $s \rightarrow 1-s$ invariance) be so everywhere.
II. Zeroes and poles

Recall that

$$\frac{1}{\Gamma(s)} = \frac{1}{s} e^{\gamma s} \prod_{n \geq 1} (1 + \frac{s}{n}) e^{-\gamma n},$$

which evidently has no poles, and no zeroes apart from those of the individual factors. So

$$\Gamma(s)$$ has no zeroes, and no poles

apart from a simple pole at each \( n \in \mathbb{Z}_{\leq 0}. \)

For the Hurwitz zeta function \( \zeta(s, x) \), we had

$$\zeta(s, x) = -\frac{1}{2\pi i} \Gamma(1-s) H_x(s),$$

with \( H_x(s) \) entire. Hence, the only possible poles are at \( m \in \mathbb{Z}_{>0} \) and are (at worst) simple poles.

But for \( m \in \mathbb{Z}_{\geq 2} \), we know \( \sum_{n \geq 0} \frac{1}{(n+m)^s} < \infty. \) So the only pole of \( \zeta(s, x) \) possible (in \( s \)) is a simple pole at \( s = 1. \) To see that it is in fact a pole, the easiest approach at this point is to
Compute

\[ H_\epsilon(1) = \int_{C_\epsilon} \frac{e^{-\epsilon w}}{1-e^{-w}} \frac{dw}{w} = \text{Res}_0 \left( \frac{e^{-\epsilon w}}{1-e^{-w}} \right) = 2\pi i. \]

For the Riemann zeta, I want to give a more direct proof:

**Lemma 1:** The function \( \zeta(s) - \frac{1}{s-1} \) extends to a holomorphic function on \( \Re(s) > 0 \). (So using the functional equation \( \zeta(s) = (2\pi)^s \Gamma(s) \frac{\sin(\frac{\pi s}{2})}{\pi} \zeta(1-s) \) to extend \( \zeta \) to \( \Re(s) < 1 \), we see that it is pole-free in that region.)

**Proof:** For \( \Re(s) > 1 \),

\[ \zeta(s) - \frac{1}{s-1} = \sum_{n=1}^{\infty} \frac{1}{n^s} - \int_{1}^{\infty} \frac{1}{x^s} \, dx \]

\[ = \sum_{n=1}^{\infty} \int_{n}^{n+1} \left( \frac{1}{x^s} - \frac{1}{(x+1)^s} \right) \, dx \]

\[ = \text{also note that } H_\epsilon(n) = 0 \text{ for } n \geq 2 \]
Where \( \left| \int_n^{n+1} \left( \frac{1}{x^s} - \frac{1}{n^s} \right) \, dx \right| \leq \left\| \frac{1}{n^s} - \frac{1}{x^s} \right\|_{[n,n+1]} \)

(since functions)

\( \lim_{n \to \infty} \left( \frac{1}{n^s} \right) = 0 \).

Take \( \frac{1}{n^s} \) to \( \frac{1}{x^s} \) in \( [n,n+1] \).

\[ \leq \frac{|s|}{n \Re(s+1)}. \]

So the sum above has absolute value bounded by

\[ |s| \sum_{n \geq 1} \frac{1}{n \Re(s+1)} \]

which converges for \( \Re(s) > 0 \).

What about the zeros of \( \zeta(s) \)? This is when the product development comes in.

**Theorem 1** For \( \Re(s) > 1 \), we have the “Euler product” expansion

\[ \zeta(s) = \prod_{p \text{ prime}} \left( 1 - p^{-s} \right)^{-1}. \]

**Proof:** Recall that for a sequence \( \{a_n\} \subseteq \mathbb{C} \)

\[ \prod (1 - a_n) \quad \text{AC} \iff \sum |a_n| \quad \text{UC}. \]

So we need \( \sum p^{-s} \quad \text{UC} \), which is clear since
\[ \sum_{n \geq 1} n^{-s} \leq \sum_{n \geq 1} n^{-(1+\varepsilon)} \left( \leq 1 + \int_{1}^{\infty} \frac{dx}{x^{1+\varepsilon}} = 1 + \frac{1}{\varepsilon} \right) \]
for \( \text{Re}(s) \geq 1 + \varepsilon \) (for any \( \varepsilon > 0 \)).

Now informally
\[
\frac{1}{1-2^{-s}} \cdot \frac{1}{1-3^{-s}} \cdot \frac{1}{1-5^{-s}} \cdot \ldots = \\
(1 + 2^{-s} + 4^{-s} + 8^{-s} + \ldots)(1 + 3^{-s} + 9^{-s} + \ldots)(1 + 5^{-s} + 25^{-s} + \ldots) \ldots = \\
1 + 2^{-s} + 3^{-s} + 4^{-s} + 5^{-s} + 6^{-s} + \ldots
\]
which works by uniqueness of factorization into primes (so that each \( n^{-s} \) occurs with multiplicity one in this product).

More formally
\[
S(s)(1 - 2^{-s}) = \sum_{n \geq 1} n^{-s} - \sum_{h \geq 1} (2n)^{-s} = \sum_{m \text{ odd}} m^{-s}
\]
\[
S(s)(1 - 2^{-s})(1 - 3^{-s}) = \sum_{m \text{ odd}} m^{-s} - \sum_{h \text{ odd}} (3m)^{-s} = \sum_{k \text{ odd} \mod 25} k^{-s} \leq 1 + \sum_{n \text{ odd}} n^{-s}
\]
\[
S(s)(1 - 2^{-s}) \ldots (1 - p_{n}^{-s}) = \sum_{k \text{ odd} \mod 25} k^{-s} \leq 1 + \sum_{n \text{ odd}} n^{-s} \xrightarrow{N \to \infty} 1
\]
(using the infinitude of primes).
Now because of the Euler product, which has no term zero for $\text{Re}(s) > 1$, $S$ has no zeroes there. For $\text{Re}(s) < 0$, use the functional equation

$$S(s) = (2\pi)^s \Gamma(1-s) \frac{\sin(\pi s/2)}{\pi} S(1-s)$$

no zeroes or poles if $\text{Re}(s) < 0$

we just showed that this has no zeroes for $\text{Re}(s) < 0$

to deduce that $S(s)$ has simple zeroes at negative even integers. So we arrive at the

**Proposition**

The zeroes of $S(s)$ are at $s \in 2\mathbb{Z}_{<0}$

and in the critical strip $0 \leq \text{Re}(s) \leq 1$. It has one (simple) pole, with residue 1, at $s = 1$.

$1,000,000$ Conjecture (Riemann Hypothesis)

The zeroes of $S(s)$ are at $2\mathbb{Z}_{<0}$ and on the line $\text{Re}(s) = 1/2$. 
Remark // By the definition of $\zeta(s)$, the fact that $\frac{n^s}{s^s} = e^{s \log n} = \sum \frac{(s \log n)^k}{k!} = \sum \frac{(s \log n)^k}{k!} = n^s$,
and the lemma, we see that $\zeta(s) = \zeta(s)$ for $\Re(s) > 0$; by the functional equation, this is clear for all $s$.
Upshot: if $\zeta$ has a zero of order $n$ at $\alpha$, then it has a zero of order $n$ at $\overline{\alpha}$.
(Also recall that there isn't one at $0$, where $\zeta(0) = -1$, or $1$, where $\zeta$ has a pole.) //

Since the $\zeta$-function is connected to primes by the Euler product expansion, one might imagine that the Riemann Hypothesis would have profound implications for their distribution. For example, one has the following result of Cramér: $\text{RH} \implies$ the gap between prime $p$ and the next prime is bounded by a constant times $\sqrt{p} \log(p)$. (We'll see another consequence when we study the Prime Number Theorem.)