Lecture 18: Primes & zeta zeroes

I. Derivative of ζ at 0

We begin with a sort of analogue of Lemma 1 of Lecture 17.

<u>Lemma A</u>: For $\mathrm{Re}(s) > -1$, we have an analytic continuation of $\zeta(s,x) = \sum_{n\geq 0} \frac{1}{(n+x)^s}$ given by

$$(*) \quad \boxed{\zeta(s,x) = \frac{x^{1-s}}{s-1} + \frac{x^{-s}}{2} - s\int_0^\infty \frac{P_1(t)}{(t+x)^{s+1}}\,dt.}$$

Proof: For $\mathrm{Re}(s) > 1$, we have (by Lecture 16, Lemma 1)

$$\sum_{k=0}^{n} \frac{1}{(k+x)^s} = \int_0^n \frac{dt}{(t+x)^s} + \frac{1}{2}\left(\frac{1}{(n+x)^s} + \frac{1}{x^s}\right) - s\int_0^n \frac{P_1(t)}{(t+x)^{s+1}}\,dt.$$

Letting $n \to \infty$, we have uniform convergence of everything for $\mathrm{Re}(s) \geq 1+\epsilon$ (for any $\epsilon > 0$), and this gives $(*)$ for <u>$\mathrm{Re}(s) > 1$.</u> But $\int_0^\infty \frac{P_1(t)}{(t+x)^{s+1}}\,dt = \int_0^\infty \frac{P_2(t)}{(t+x)^{s+2}}\,dt$ (argue as in Lecture 16 lemma 2) is uniformly convergent for $\mathrm{Re}(s) \geq -1+\epsilon$ ($\forall \epsilon > 0$) and so yields an analytic function there, giving via $(*)$ the analytic continuation of $\zeta(s,x)$. $\square$

This allows us to perform a power-series expansion of ζ about 0:

<u>**Lemma B**</u>: $\zeta(s,x) = -\frac{1}{2} - x - (\log D(x))s + O(s^2)$,

where $D(x) := \dfrac{\sqrt{2\pi}}{\Gamma(x)}$ and "$O(s^2)$" means a function f with $|f(s)| \leq C|s|^2$ (for some $C>0$) for $|s|$ suff. small.

Proof:

- $x^{-s} = e^{-s\log x} = 1 - s\log x + O(s^2)$
- $\dfrac{x^{1-s}}{s-1} = x \cdot \dfrac{-1}{1-s} \cdot x^{-s} = -x(1-s\log x)(1+s) + O(s^2)$
- $-s\displaystyle\int_0^\infty \frac{P_1(t)}{(t+x)^{s+1}}\,dt$ is holomorphic at $s=0$, and at $s=0$ the integral is equal to $\displaystyle\int_0^\infty \frac{P_1(t)}{t+x}\,dt$

So putting everything together gives

$$\zeta(s,x) = -x(1-s\log x)(1+s) + \frac{1-s\log x}{2} - s\int_0^\infty \frac{P_1(t)}{t+x}\,dt + O(s^2)$$

$$= \frac{1}{2} - x + s\underbrace{\left\{\left(x-\frac{1}{2}\right)\log x - x - \int_0^\infty \frac{P_1(t)}{t+x}\,dt\right\}}_{\substack{\text{Stirling:}\quad = \log\Gamma(x) - \frac{1}{2}\log 2\pi \\ = -\log D(x)}} + O(s^2)$$

$\square$

One nice application of this is the

Theorem 2 (Lerch formula) $\log D(x) = -\zeta'(0, x)$. ← w.r.t. s

Proof: Obvious from Lemma B. (Note the special case $\zeta'(0) = -\frac{1}{2}\log 2\pi$.) $\square$

For Riemann zeta, one also has a nice Laurent-series expansion at $s = 1$:

Proposition $\zeta(s) = \dfrac{1}{s-1} + \gamma + O(s-1)$.

Proof: In lemma 1 of Lect. 17, we have (substituting $x = 1$)

$$\zeta(s) = \frac{1}{s-1} + \frac{1}{2} - s\int_0^\infty \frac{P_1(t)}{(1+t)^{s+1}}\,dt$$

$$= \frac{1}{s-1} + \frac{1}{2} - (s-1)\underbrace{\int_0^\infty \frac{P_1(t)}{(1+t)^{s+1}}\,dt}_{\text{holo. at } s=1} - \underbrace{\int_0^\infty \frac{P_1(t)}{(1+t)^{s+1}}\,dt}_{\text{holo. at } s=1}$$

$$= \frac{1}{s-1} + \frac{1}{2} - \int_0^\infty \frac{P_1(t)}{(1+t)^2}\,dt + O(s-1).$$

Now by Lecture 16 (Lemma 1),

$$1 + \frac{1}{2} + \dots + \frac{1}{n+1} = \int_0^n \frac{dt}{1+t} + \frac{1}{2}\left(\frac{1}{1+n} + \frac{1}{1}\right) - \int_0^n \frac{P_1(t)}{(1+t)^2}\,dt$$

$$\Rightarrow 1 + \frac{1}{2} + \dots + \frac{1}{n+1} - \log(1+n) = \frac{2+n}{2+2n} - \int_0^n \frac{P_1(t)}{(1+t)^2}\,dt$$

$$\xRightarrow{\left(\lim_{n\to\infty}\right)} \quad \gamma = \frac{1}{2} - \int_0^\infty \frac{P_1(t)}{(1+t)^2}\,dt,$$

completing the proof. $\square$

$\boxed{\text{II. A counting function}}$

Recall summation by parts:

$$\sum_{k=0}^{n} a_k b_k = a_n B_n - \sum_{k=0}^{n-1} B_k (a_{k+1} - a_k) \qquad \left(B_\ell = \sum_{i=0}^{\ell} b_i \right)$$

Set $a_0 = 0$, $a_1 = 0$, $a_k = \log 2 + \cdots + \log p_{k-1}$

$b_0 = -1$, $b_k = \frac{1}{p_{k-1}^s} - \frac{1}{p_k^s}$ (where $p_0 = 1$, $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, ... is an enumeration of the primes).

Then

$$\sum_{k=\cancel{0}\,1}^{n} a_k b_k = \underbrace{a_n \cdot \frac{-1}{p_n^s}} - \sum_{k=1}^{n-1} \left(-\frac{1}{p_k^s} \right) \log p_k$$

$$|\cdot| \le \left| \frac{n \log p_n}{p_n^s} \right| \le \left| \frac{\cancel{p_n} \log p_n}{p_n^{\cancel{s}\, s-1}} \right| \xrightarrow[n \to \infty]{} 0$$

(for $\mathrm{Re}(s) > 1$)

So

$$(\#) \qquad \sum_{k=1}^{\infty} a_k b_k = \sum_{k=1}^{\infty} \frac{\log p_k}{p_k^s}.$$

Define

$$\varphi(x) := \sum_{\substack{p \le x \\ p \text{ prime } (>1)}} \log(p).$$

We are ultimately interested in proving $\varphi(x) \sim x$ for $x \to \infty$, but for now will investigate the related function

$$\Phi(s) := s\int_1^\infty \frac{\varphi(x)}{x^{s+1}}\,dx \qquad (\mathrm{Re}(s)>1)$$

$$= s\sum_{k=1}^\infty \int_{p_{k-1}}^{p_k} \frac{\varphi(x)}{x^{s+1}}\,dx \quad \longleftarrow = a_k \text{ on this interval}$$

$$= \cancel{s}\sum_{k=1}^\infty \frac{a_k}{\cancel{s}}\underbrace{\left(\frac{1}{p_{k-1}^s}-\frac{1}{p_k^s}\right)}_{b_k}$$

$$\overset{\text{by }(\#)}{=} \sum_{k=1}^\infty \frac{\log p_k}{p_k^s}$$

$$= \sum_{p \text{ prime} > 1}\left(\frac{\log p}{p^s}\right),$$

which is dominated for $\mathrm{Re}(s) \geq 1+\epsilon$ by

$$\sum \frac{\log n}{n^{1+\epsilon}} \leq \int_1^\infty \frac{\log x}{x^{1+\epsilon}}\,dx \underset{(\int \text{ by parts})}{=} \frac{1}{\epsilon^2}$$

hence is AC & UC there $(\forall \epsilon > 0)$.

$\boxed{\text{Theorem 3}}$ (a) Φ is meromorphic for $\mathrm{Re}(s) > \frac{1}{2}$

(b) For $\mathrm{Re}(s) \geq 1$, $\begin{cases} \zeta(s) \neq 0 & \text{and} \\ \Phi(s) - \frac{1}{s-1} \neq \infty \end{cases}$.

Proof: Concerning (b), we know for $\mathrm{Re}(s)>1$ that

$\zeta(s) \neq 0$. From $\zeta(s)^{-1} = \prod_{p \text{ prime}} (1-p^{-s})$ we get for $\mathrm{Re}(s) > 1$

$$-\frac{\zeta'}{\zeta}(s) = \sum_p \frac{d}{ds}\log(1-p^{-s})$$
$$= \sum_p \frac{(\log p)\,p^{-s}}{1-p^{-s}}$$
$$= \sum_p \frac{\log p}{p^s - 1}$$

$$\frac{1}{p^s-1} = \frac{1}{p^s}\frac{1}{1-p^{-s}} = \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots$$

$$= \sum_p h_p(s) + \Phi(s),$$

where $|h_p(s)| \le C\dfrac{\log p}{|p^{2s}|}$. Now, $\sum \dfrac{\log n}{n^{2s}}$ is AC & UC for $\mathrm{Re}(s) \ge \frac{1}{2}+\delta$ $(\delta > 0)$. By Lec. 17 Lemma 1 (holomorphicity of $\zeta(s) - \frac{1}{s-1}$ for $\mathrm{Re}(s) > 0$), we conclude that Φ is meromorphic on $\mathrm{Re}(s) > \frac{1}{2}$ with poles at $s=1$ and the zeroes of ζ (which can occur only for $\mathrm{Re}(s) \le 1$).

To show ζ has no zeroes on $\mathrm{Re}(s) = 1$ [which implies (by the functional equation) the same for $\mathrm{Re}(s) = 0$], consider

$$\zeta(s) = \prod_p \frac{1}{1-p^{-s}} = \prod_p e^{-\log(1-p^{-s})}$$
$$= \exp\left(\sum_p \sum_{m\ge 1} \frac{1}{mp^{ms}}\right)$$

$$\Longrightarrow \quad |\zeta(s)| = \exp\left(\sum_p \sum_{m\geq 1} \frac{1}{m} \operatorname{Re}(p^{-ms})\right)$$

$s = \sigma + it$

$e^{-ms\log p} = e^{-m\sigma\log p} e^{-imt\log p}$

$$= \exp\left(\sum_p \sum_m \frac{1}{m} e^{-m\sigma\log p} \cos(mt\log p)\right)$$

$$= \exp\left(\sum_p \sum_m \frac{\cos(mt\log p)}{m p^{m\sigma}}\right)$$

$$\Longrightarrow \quad \zeta^3(\sigma)\,|\zeta(\sigma+it)|^4\,|\zeta(\sigma+2it)|$$

$$= \exp\left(\sum_p \sum_m \frac{3 + 4\cos(mt\log p) + \cos(2mt\log p)}{m p^{m\sigma}}\right)$$

$$\Longrightarrow \quad \boxed{\zeta^3(\sigma)\,|\zeta(\sigma+it)|^4\,|\zeta(\sigma+2it)| \geq 1 \quad (\text{for } \sigma > 1)} \qquad (\#\#)$$

$$\left[\begin{array}{l} 2\cos^2\theta = \cos 2\theta + 1 \\ \quad \Downarrow \\ 3 + 4\cos\theta + \cos 2\theta = 2(1+\cos\theta)^2 \geq 0 \end{array}\right]$$

Suppose $\zeta(1+it_0) = 0$ for some $t_0 \neq 0$: then

$\zeta(\sigma + it_0) = O(\sigma - 1) \Rightarrow |\zeta(\sigma+it_0)|^4 = O((\sigma-1)^4)$,

while $\zeta(\sigma) = \Theta\left(\frac{1}{\sigma-1}\right) \Rightarrow \zeta^3(\sigma) = \Theta\left(\frac{1}{(\sigma-1)^3}\right)$

and $|\zeta(\sigma + 2it_0)| = \Theta(1)$ (is bounded)

$\Rightarrow$ LHS(##) $= O(\sigma - 1)$, contradicting the inequality! $\square$

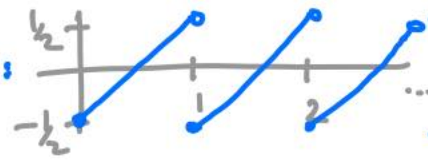
Appendix: for your reference

LECTURE 16

Lemma 1: Let $f : [0,n] \to \mathbb{R}$ be C^1.

Then $$\sum_{k=0}^{n} f(k) = \int_0^n f(t)\,dt + \frac{f(n)+f(0)}{2} + \int_0^n P_1(t) f'(t)\,dt,$$

where $P_1(t) := t - \lfloor t \rfloor - \frac{1}{2}$. [Picture:



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Lemma 2: $$\tilde{\mathcal{J}}(z) := \int_0^\infty \frac{P_1(t)}{z+t}\,dt \in \mathrm{Hol}(\mathbb{C} \setminus \mathbb{R}_{\le 0}).$$

LECTURE 17

Lemma 1: The function $\zeta(s) - \frac{1}{s-1}$ extends to a holomorphic function on $\mathrm{Re}(s) > 0$.

Key point in proof: for $\mathrm{Re}(s) > 1$,

$$\zeta(s) - \frac{1}{s-1} = \sum_{n \ge 1} \frac{1}{n^s} - \int_1^\infty \frac{1}{x^s}\,dx$$

and this converges for $\mathrm{Re}(s) > 0$

$$= \sum_{n \ge 1} \int_n^{n+1} \left(\frac{1}{n^s} - \frac{1}{x^s} \right) dx \qquad (*)$$