Lecture 19: The prime number theorem

I. A Laplace Tauberian theorem

Before resuming our investigation of the counting function \( p \), we will need an integral analogue of Tauber's theorem. Recall that the idea of a Tauberian theorem for series was:

If \( f(s) = \sum a_n s^n \) has radius of convergence 1 (and is considered to be defined on \( D_1 \)) and
\[
\lim_{x \to 1^-} f(x) \text{ exists, then provided some regularity assumption holds for } \{a_n\} \text{ (e.g. } |a_n| \text{ bounded, } n a_n \to 0 \text{, etc.), } \sum a_n \text{ exists and equals that limit.}
\]

The integral analogue is
**Theorem 1** Let \( f: (0, \infty) \to \mathbb{C} \) be bounded and piecewise \( C^0 \), and set \( U_0 := \{ z \in \mathbb{C} \mid \text{Re}(z) > 0 \} \).

(Laplace transformation) \( g(z) := \int_0^{\infty} f(t) e^{-zt} \, dt \in \text{Hol}(U_0) \).

If there exists \( \tilde{g} \in \text{Hol}(U_0) \) extending \( g \), then \( \int_0^{\infty} f(t) \, dt \) exists and equals \( \tilde{g}(0) \).

First, why is \( g \in \text{Hol}(U_0) \)? This is by the Lemma of Lemma 15, using the uniform convergence of \( \int_0^{\infty} f(t) e^{-zt} \, dt \) on sectors \( U_{\epsilon} \) (\( \epsilon > 0 \)) and analyticity of the argument in \( z \).

By the same token, for any \( T > 0 \) the function \( g_T(z) := \int_0^{T} f(t) e^{-zt} \, dt \) is entire.

Now since \( \tilde{g} \in \text{Hol}(U_0) \), it is holomorphic on an open set containing \( \overline{U}_0 \): call this \( \tilde{U}_0 \). Given any \( R > 0 \), compactness of \( [-R, R] \) \( \Rightarrow \)
\[ \Delta_R := \min \left( \max_{y \in [-R,R]} \{ r : D(iy, r) \subset U_0 \} \right) > 0. \]

Taking \( \delta \in (0, \Delta R) \), \( \tilde{g} \) is defined on a region containing

\[ \gamma := \gamma_+ + \gamma_- \]

and the region it encloses. By Cauchy (\( e^{\gamma e^T(1+\frac{\delta^2}{R^2})} = 1 \)),

\[ \tilde{g}(0) - g_T(0) = \frac{1}{2\pi i} \int_{\gamma} \left( (\tilde{g}(z) - g_T(z)) e^{T_z(1+\frac{\delta^2}{R^2})} \right) \frac{dz}{z} \]

\[ = \frac{1}{2\pi i} \int_{\gamma} H_T(z) \, dz. \]

Next, recall that \( \|f\|_{\ell_2} \leq B \) by assumption.

**Lemma 1:** \[ \left| \frac{1}{2\pi i} \int_{\gamma} H_T(z) \, dz \right| \leq \frac{B}{R}. \]

**Proof:** For \( z \in U_0 \),

\[ |g_T(z) - g_T(\theta)| = \left| \int_T g(\theta) e^{-e^{\theta} \, d\theta} \right| \leq B \int_T |e^{-\theta}| \, d\theta = \frac{B}{e^{-Re(\theta)}}. \]

For \( |\Re| = R \),

\[ |e^{\theta T(1+\frac{\delta^2}{R^2})} \frac{1}{T} | = \frac{e^{Re(\theta)T}}{R} \cdot | \frac{R + \frac{\delta^2}{R} }{T} | \]
\[
\mathcal{N} = e^{i\alpha T} \frac{2|\text{Re}(e)|}{R^2}. \quad \text{So for } \theta \in \{e^{+}\}, \quad \text{or } \theta \in \{e^{-}\},
\]

\[
\left| H_+ (\theta) \right| \leq \frac{B}{\text{Re}(e)} e^{-\text{Re}(e) T} \cdot e^{\text{Re}(e) T} \frac{2|\text{Re}(e)|}{R^2} = \frac{2B}{R^2}
\]

\[
\text{and } \frac{1}{2\pi} \int_{\gamma^+} H_+ (\theta) \, d\theta \leq \frac{1}{2\pi} L(\gamma^+) \frac{2B}{R^2} = \frac{B}{R}.
\]

**Lemma 2:** \[ \left| \frac{1}{2\pi i} \int_{\gamma^-} g_+ (\theta) e^{T \theta} (1 + \frac{\theta^2}{R^2}) \, d\theta \right| \leq \frac{B}{R}. \]

**Proof:** Write \( C_- \) for \( \{ |z| = R \} \cap \{ \text{Re}(\theta) \leq 0 \} \), so that \( \gamma_- - C_- \) is closed. As the integrand only has a pole at \( \theta = 0 \), \[ \int_{\gamma_- - C_-} g_+ (\theta) e^{T \theta} (1 + \frac{\theta^2}{R^2}) \, d\theta = 0. \]

For \( \theta \in C_- \), \[ |g_+ (\theta)| = \left| \int_0^T f(t) e^{-t \theta} \, dt \right| \leq B \int_0^T e^{-\text{Re}(\theta) t} \, dt = B \left\{ \frac{1}{\text{Re}(\theta)} - e^{-\text{Re}(\theta) T} \right\} \leq B e^{-\text{Re}(\theta) T} / |\text{Re}(\theta)| \]

and so

\[
\frac{1}{2\pi} \left| \int_{C_-} g_+ (\theta) e^{T \theta} (1 + \frac{\theta^2}{R^2}) \, d\theta \right| \leq \frac{1}{2\pi} \frac{B e^{-\text{Re}(\theta) T} / |\text{Re}(\theta)|}{2\pi} \frac{2|\text{Re}(\theta)|}{R^2} \leq B / R.
\]

**Lemma 3:** \[ \lim_{T \to \infty} \int_{\gamma^-} g_+ (\theta) e^{T \theta} (1 + \frac{\theta^2}{R^2}) \, d\theta = 0. \]

**Proof:** Set \( h(\theta) := g_+ (\theta) (1 + \frac{\theta^2}{R^2}) \frac{i}{2} \); the integral is
\[ \int_{\gamma_0} h(t) e^{T_0 t} \, dt, \quad \text{where } h(t) \text{ is analytic on a neighborhood of } \gamma. \] By compactness of \( \gamma \), \( \|h\|_{\gamma} \leq M \), and so on \( \gamma \) we have \( |h(t) e^{T_0 t}| \leq Me^{-T_0 t} \). Subdividing \( \gamma = \gamma_0 + \gamma_1 \) (see figure), we have (uniformly) \( \|h(t) e^{T_0 t}\|_{\gamma_0} \leq Me^{-T_0 \delta} \to 0 \), so that the \( \int_{\gamma_0} \) part of the integral does limit to 0. On the remaining small part \( \gamma_1 \), the integrand is bounded by \( M \).

So given \( \epsilon > 0 \), if we take \( \delta_1 < \frac{\epsilon}{8M} \) and
\[ T > \frac{1}{\delta_1} \log \left( \frac{2M L(\gamma)}{\epsilon} \right), \quad \text{then} \quad \left| \int_{\gamma_0} h(t) e^{T_0 t} \, dt \right| \leq \int_{\gamma_0} Me^{-T_0 \delta_1} \, dt + \int_{\gamma_1} M \, dt < L(\gamma)Me^{-T_0 \delta_1} + M \cdot 4 \cdot \frac{\epsilon}{8M} = \frac{\epsilon}{8} + \frac{\epsilon}{2} = \epsilon. \]

**Proof of Theorem 1:** Given \( \epsilon > 0 \), pick \( R \) s.t. \( \frac{2R}{R} < \frac{\epsilon}{2} \) and (by Lemma 3) \( T \) s.t. \( \left| \int_{\gamma_0} \frac{\partial}{\partial t} g(t) e^{T_0 t} (\Delta t^2) \, dt \right| < \frac{\epsilon}{2} \).

Then \( \left| g(0) - g_T(0) \right| = \left| \frac{1}{2R} \int_{\gamma_0} H(t) \, dt \right| \leq \frac{B}{R} + \frac{B}{R} + \frac{\epsilon}{2} < \epsilon \),

which proves that

\[ \text{Lemma 1, Lemma 2} \]
\[ \lim_{T \to \infty} g_T(0) \text{ exists and equals } \dot{\gamma}(0). \]

But also, \[ \lim_{T \to \infty} g_T(0) = \lim_{T \to \infty} \int_0^T f(t)e^{-0.5t} \, dt \]

(by defn. !)

II. Proof of the prime number theorem

Back to our "counting functions". Recall that

\[ \Psi(x) := \sum_{p \leq x} \log p \]

and

\[ \Phi(x) := \sum_{x}^{\infty} \frac{\psi(x)}{x^{s+1}} \, dx \quad (\Re(s) > 1) \]

\[ = \sum_{\rho \text{ prime}} \frac{\log p}{p^s} \]

extends to a meromorphic function on \( \Re(s) > \frac{1}{2} \) with poles at \( s = 1 \) (with residue 1) and at the zeros of the Riemann zeta function (which only occur, if at all, on \( \frac{1}{2} < \Re(s) < 1 \) in this region).

**Theorem 2**

\[ \psi(x) = \Theta(x) \quad (\text{that is, } \exists C > 0 \text{ and } M > 0 \text{ s.t. } |\psi(x)| \leq C x \quad \forall \ x \geq M). \]
Proof: For \( n \in \mathbb{N} \),

\[
2^{2n} = (1 + 1)^{2n} = \sum_{j=0}^{2n} \binom{2n}{j} \geq \binom{2n}{n} \geq \prod_{n < p \leq 2n} p = e^{\varphi(2n)} - e^{\varphi(n)}
\]

Take \( \log \) to get

\[
\varphi(2n) - \varphi(n) \leq 2n \log 2
\]

\[
\Rightarrow \quad \varphi(2^k) = \sum_{k=1}^{k} (\varphi(2^k) - \varphi(2^{k-1})) \quad \left\{ \varphi(1) = 0 \right\}
\]

\[
\leq \sum_{k=1}^{k} 2^k \log 2
\]

\[
< 2^{k+1} \log 2.
\]

Given \( x > 1 \), \( 2^{k-1} < x < 2^k \) for some \( k \) and

\[
\varphi(x) \leq \varphi(2^k) < 2^{k+1} \log 2 < (4 \log 2) \cdot x.
\]

We will also need the following consequence of Theorems 1 & 2:

**Proposition** \( \int_1^\infty \frac{\varphi(x) - x}{x^2} \, dx \) converges!

Proof: Set \( f(t) := \varphi(e^t)e^{-t} - 1 \) (piecewise \( C^x \)

on \( [0, \infty) \)). By Theorem 2, \( |\varphi(e^t)e^{-t}| < Ce^t e^{-t} = C \)

on \( [0, \infty) \). So Theorem 1 applies, and we have that
\[
\int_0^\infty f(t) \, dt = \int_1^{e^t} \frac{g(x) - x}{x} \, dx \quad \text{converges provided}
\]
\[
(x = e^t, \quad \log x = t)
\]

that the Laplace transform
\[
g(t) = \int_0^\infty f(t) e^{-zt} \, dt \in \mathcal{H}(U_0)
\]
extends to \( \tilde{g} \in \mathcal{H}(\overline{U}_0) \). For \( Re(z) > 0 \),
\[
g(t) = \int_0^\infty \frac{g(e^t) - e^t}{e^t} e^{-zt} \, dt
\]
\[
= \int_0^\infty \frac{g(e^t) - e^t}{e^{2t}} e^{-zt} e^{zt} \, dt
\]
\[
= \int_0^\infty \left( \frac{\Phi(x) - x}{x^{2+1}} \right) \, dx
\]
\[
(\log x = t)
\]
\[
= \frac{\Phi(t+1)}{t+1} - \frac{1}{2}
\]

which extends to a meromorphic function on \( Re(z) > -\frac{1}{2} \) with no poles on the line \( Re(z) = 0 \). (This is Theorem 3 of Lecture 18, with \( s = t+1 \).) So our \( \tilde{U} \) is just \( \{Re(z) > -\frac{1}{2}\} \setminus \{\text{poles} \} (\supset \overline{U}_0) \), and we have met the conditions of Theorem 1.

**Theorem 3** \( \Phi(x) \sim x \), i.e. \( \lim_{x \to \infty} \frac{\Phi(x)}{x} = 1 \).
**Proof:** We must show \( \lim_{x \to \infty} \frac{\varphi(x)}{x} \leq 1 \) and
\[
\liminf_{x \to \infty} \frac{\varphi(x)}{x} \geq 1.
\]
If the first fails, \( \exists \delta > 0 \) such for certain arbitrarily large numbers \( y \), \( \varphi(y) > (1 + 2\delta)y \), hence \( \varphi(y) > (1 + \delta)x \) for \( y < x < \rho y \), where \( \rho := \frac{1 + 2\delta}{1 + \delta} \). But then
\[
\int_y^{\rho y} \frac{\varphi(x) - x}{x^2} \, dx \geq \int_y^{\rho y} \frac{\delta}{x} \, dx = \delta \log(\rho)
\]
\[
(\rho(x) > \varphi(y) > (1 + \delta)x)
\]
for those same numbers \( y \), contradicting the existence of the integral in the Proposition.

**Exercise:** Show that we cannot have \( \liminf_{x \to \infty} \frac{\varphi(x)}{x} < 1 \)
by the same approach, by considering intervals \( \theta y < x < y \)
with \( \theta < 1 \), where \( \varphi(\theta y) < (1 + \delta)y \).

This completes the proof.

Now let
\[\pi(x) := \text{number of primes } \leq x\]
Proof: We have

\[ \Phi(x) = \sum_{p \leq x} \log p \leq \sum_{p \text{ prime}} \log x = \pi(x) \log x \]

\[ \Rightarrow \pi(x) \cdot \frac{\log x}{x} \geq \frac{\Phi(x)}{x} \quad \text{(1)} \]

Further, for \( 1 < y < x \),

\[ \pi(x) = \pi(y) + \sum_{y < p \leq x} 1 \leq \pi(y) + \sum_{y < p \leq x} \frac{\log p}{\log y} \]

\[ \Rightarrow \pi(x) < y + \frac{\Phi(x)}{\log y} \quad \text{(2 \# \#)} \]

Taking \( y = \frac{x}{\log^2 x} \) in (2 \# \#),

\[ \pi(x) \frac{\log x}{x} < \left( \frac{x}{\log^2 x} + \frac{\Phi(x)}{\log x - 2 \log(\log x)} \right) \frac{\log x}{x} \]

\[ = \frac{1}{\log x} + \frac{\Phi(x)}{x} \cdot \frac{\log x}{\log x - 2 \log(\log x)} \]

Which, together with (1) gives
\[
\frac{\varphi(x)}{x} \leq \frac{\varphi(x)}{x} \frac{\log x}{x} \leq \frac{1}{\log x} + \frac{\varphi(x)}{x} \frac{\log x}{\log x - 2 \log (\log x)}.
\]

Whereupon Theorem 3 applies to give the result.

Remark: The function
\[
\text{Li}(x) := \int_0^x \frac{dt}{\log t}
\]
is actually known to do a better job at approximating \(\pi(x)\) than \(\pi(x) / \log(x)\). By a result of Schoenfield, if the Riemann hypothesis holds then one has
\[
|\pi(x) - \text{Li}(x)| \leq \frac{1}{8\pi} \sqrt{x} \log x \quad \text{for all } x \geq 26.57.
\]