Lecture 19 : The prime number theorem

I. A Laplace Tauberian theorem

Before resuming our investigation of the counting function φ, we will need an integral analogue of Tauber's theorem. Recall that the idea of a Tauberian theorem for series was:

If $f(z) = \sum a_n z^n$ has radius of convergence 1 (and is considered to be defined on D_1) and $\lim_{x \to 1^-} f(x)$ exists, then provided some regularity assumption holds for $\{a_n\}$ (e.g. $|a_n|$ bounded, $na_n \to 0$, etc.), $\sum a_n$ exists and equals that limit.

The integral analogue is

$\boxed{\text{Theorem 1}}$ Let $f : [0, \infty) \to \mathbb{C}$ be bounded and piecewise C^0, and set $U_0 := \{ z \mid Re(z) > 0 \}$,

(Laplace transform) $g(z) := \int_0^\infty f(t) e^{-zt} dt \in Hol(U_0)$.

If there exists $\tilde{g} \in Hol(\overline{U_0})$ extending g, then $\int_0^\infty f(t) dt$ exists and equals $\tilde{g}(0)$.

First, why is $g \in Hol(U_0)$? This is by the Lemma of Lecture 15, using the uniform convergence of $\int_0^\infty f(t) e^{-zt} dt$ on subsets $\overline{U_\epsilon}$ $(\epsilon > 0)$ and analyticity of the argument in z.

By the same token, for any $T > 0$ the function

$$g_T(z) := \int_0^T f(t) e^{-zt} dt$$

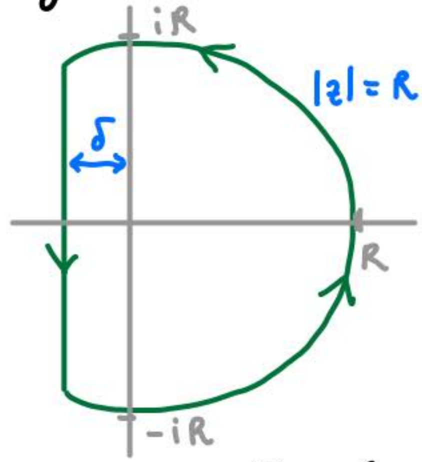
is <u>entire</u>.

Now since $\tilde{g} \in Hol(\overline{U_0})$, it is holomorphic on an open set containing $\overline{U_0}$: call this $\tilde{U}_0$. Given any $R > 0$, compactness of $[-R, R] \Longrightarrow$

$$\Delta_R := \min_{y\in[-R,R]}\left(\max\left\{r \,\middle|\, D(iy,r)\subset \tilde{U}_0\right\}\right) > 0 .$$

Taking $\delta \in (0,\Delta_R)$, $\tilde{g}$ is defined on a region containing

$\gamma :=$



$= \gamma_+ + \gamma_-$

part with $Re(z) \le 0$ (γ_-)

part with $Re(z) \ge 0$ (γ_+)

and the region it encloses. By Cauchy ($\because e^{T\cdot 0}(1+\frac{0^2}{R^2}) = 1$),

$$\tilde{g}(0) - g_T(0) = \frac{1}{2\pi i}\int_\gamma \left(\tilde{g}(z)-g_T(z)\right)e^{Tz}\left(1+\frac{z^2}{R^2}\right)\frac{dz}{z}$$

$$= \frac{1}{2\pi i}\int_\gamma H_T(z)\,dz .$$

Next, recall that $\|f\|_{[0,\infty)} \le B$ by assumption.

<u>Lemma 1</u>: $\left|\frac{1}{2\pi i}\int_{\gamma_+} H_T(z)\,dz\right| \le \frac{B}{R}$.

Proof: For $z\in U_0$, $|g(z)-g_T(z)| = \left|\int_T^\infty f(t)e^{-zt}\,dt\right|$

$$\le B\int_T^\infty \underbrace{\left|e^{-zt}\right|}_{=e^{-Re(z)t}}dt = \frac{B}{Re(z)}e^{-Re(z)T}.$$

For $|z| = R$, $\left|e^{zT}\left(1+\frac{z^2}{R^2}\right)\frac{1}{z}\right| = \frac{e^{Re(z)T}}{R}\cdot\left|\frac{R}{z}+\frac{z}{R}\right|$

$= e^{\mathrm{Re}(z)T} \cdot \dfrac{2|\mathrm{Re}(z)|}{R^2}$. So for $z \in \gamma_+$,

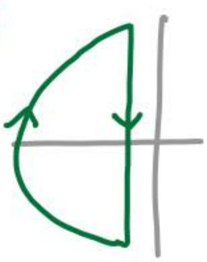
$\left(\dfrac{R}{Re^{i\theta}} + \dfrac{Re^{i\theta}}{R} = e^{-i\theta} + e^{i\theta} = 2\cos\theta = \dfrac{2}{R}\mathrm{Re}(z)\right)$

$$|H_T(z)| \le \frac{B}{\cancel{\mathrm{Re}(z)}} \cancel{e^{-\mathrm{Re}(z)T}} \cdot \cancel{e^{\mathrm{Re}(z)T}} \, \frac{2\cancel{\mathrm{Re}(z)}}{R^2} = \frac{2B}{R^2}$$

and $\dfrac{1}{2\pi}\left|\int_{\gamma_+} H_T(z)\,dz\right| \le \dfrac{1}{2\pi} \overbrace{L(\gamma_+)}^{\pi R} \dfrac{2B}{R^2} = \dfrac{B}{R}$. □

Lemma 2: $\left|\dfrac{1}{2\pi i}\int_{\gamma_-} g_T(z)\, e^{Tz}\left(1+\dfrac{z^2}{R^2}\right)\dfrac{dz}{z}\right| \le \dfrac{B}{R}$.

Proof: Write C_- for $\{|z| = R\} \cap \{\mathrm{Re}(z) \le 0\}$, so that $\gamma_- - C_-$ is closed. As the integrand only has a pole at $z=0$, $\int_{\gamma_- - C_-} g_T(z)\, e^{Tz}\left(1+\frac{z^2}{R^2}\right)\frac{dz}{z} = 0$.



For $z \in C_-$, $|g_T(z)| = \left|\int_0^T f(t) e^{-tz}\,dt\right|$

$$\le B\int_0^T e^{-\mathrm{Re}(z)t}\,dt = B\left\{\underbrace{\frac{1}{\mathrm{Re}(z)}}_{<0} - \frac{e^{-\mathrm{Re}(z)T}}{\mathrm{Re}(z)}\right\} \le B\, e^{-\mathrm{Re}(z)T}/|\mathrm{Re}(z)|$$

and so

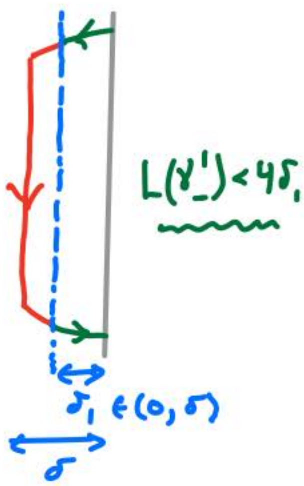
$$\frac{1}{2\pi}\left|\int_{C_-} g_T(z)\, e^{Tz}\left(1+\frac{z^2}{R^2}\right)\frac{dz}{z}\right| \le \cancel{\frac{\pi R}{2\pi}}\, B\, \frac{\cancel{e^{-\mathrm{Re}(z)T}}}{\cancel{|\mathrm{Re}(z)|}} \cdot \cancel{e^{\mathrm{Re}(z)T}}\, \frac{\cancel{2|\mathrm{Re}(z)|}}{R^{\cancel{2}}} = B/R.$$

□

Lemma 3: $\lim\limits_{T\to\infty} \int_{\gamma_-} \tilde{g}(z)\, e^{Tz}\left(1+\dfrac{z^2}{R^2}\right)\dfrac{dz}{z} = 0$.

Proof: Set $h(z) := \tilde{g}(z)\left(1+\dfrac{z^2}{R^2}\right)\dfrac{1}{z}$; the integral is

$\int_{\gamma_-} h(z)\, e^{Tz}\, dz$, where $h(z)$ is analytic on a neighborhood of γ_-. By compactness of γ_-, $\|h\|_{\gamma_-} \le M$, and so on γ_- we have $|h(z)e^{Tz}| \le M e^{\mathrm{Re}(z)T}$.



Subdividing $\gamma_- = \gamma_-^0 + \gamma_-^1$ (see figure), we have (uniformly) $\|h(z)e^{Tz}\|_{\gamma_-^0} \le M e^{-T\delta_1} \to 0$, so that the $\int_{\gamma_-^0}$ part of the integral does limit to 0. On the remaining small part γ_-^1, the integrand is bounded by M.

So given $\epsilon > 0$, if we take $\delta_1 < \frac{\epsilon}{8M}$ and $T > \frac{1}{\delta_1}\log\left(\frac{2ML(\gamma_-)}{\epsilon}\right)$, then $\left|\int_{\gamma_-} h(z)e^{Tz}\,dz\right| \le$

$$\int_{\gamma_-^0} M e^{-T_0\delta_1}\,dz + \int_{\gamma_-^1} M\,dz < L(\gamma_-)\cdot M e^{\log\left(\frac{\epsilon}{2ML(\gamma_-)}\right)} + M\cdot 4\cdot\frac{\epsilon}{8M}$$

$$= \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \quad \square$$

Proof of Theorem 1: Given $\epsilon > 0$, pick R s.t. $\frac{2B}{R} < \frac{\epsilon}{2}$ and (by Lemma 3) T s.t. $\left|\int_{\gamma_-} \tilde{g}(z)e^{Tz}\left(1+\frac{z^2}{R^2}\right)\frac{dz}{z}\right| < \frac{\epsilon}{2}$.

Then $|\tilde{g}(0) - g_T(0)| = \left|\frac{1}{2\pi i}\int_\gamma H_T(z)\,dz\right| \le \underbrace{\frac{B}{R}}_{\text{Lemma 1}} + \underbrace{\frac{B}{R}}_{\text{Lemma 2}} + \frac{\epsilon}{2} < \epsilon$,

which proves that

$\lim_{T\to\infty} g_T(0)$ exists and equals $\tilde{g}(0)$.

But also, $\lim_{T\to\infty} g_T(0) = \lim_{T\to\infty}\int_0^T f(t)\cancel{e^{-0\cdot t}}\,dt$ (by defn.!)

$= \int_0^\infty f(t)\,dt.$ $\square$

II. Proof of the prime number theorem

Back to our "counting functions". Recall that

$$\varphi(x) := \sum_{\substack{p\le x\\ p \text{ prime}}} \log(p)$$

and

$$\Phi(s) := s\int_1^\infty \frac{\varphi(x)}{x^{s+1}}\,dx \qquad (\mathrm{Re}(s)>1)$$

$$= \sum_{p \text{ prime}} \frac{\log p}{p^s}$$

extends to a meromorphic function on $\mathrm{Re}(s) > \frac{1}{2}$ with poles at $s=1$ (with residue 1) and at the zeroes of the Riemann zeta function (which only occur, if at all, on $\frac{1}{2}<\mathrm{Re}(s)<1$ in this region).

$\boxed{\text{Theorem 2}}$ $\varphi(x) \overset{(x\to\infty)}{=} O(x)$ (that is, $\exists\, C>0$ and $M>0$ s.t. $|\varphi(x)| \le Cx \;\; \forall\, x \ge M$).

Proof: For $n \in \mathbb{N}$,

$$2^{2n} = (1+1)^{2n} = \sum_{j=0}^{2n} \binom{2n}{j} \geq \binom{2n}{n} \geq \prod_{\substack{p \text{ primes,} \\ n < p \leq 2n}} p = e^{\varphi(2n) - \varphi(n)}$$

$$\xRightarrow[\text{log}]{\text{take}} \varphi(2n) - \varphi(n) \leq 2n \log 2$$

$$\Longrightarrow \varphi(2^k) = \sum_{\ell=1}^{k} \left(\varphi(2^\ell) - \varphi(2^{\ell-1})\right) \qquad [\varphi(1) = 0]$$

$$\leq \sum_{\ell=1}^{k} 2^\ell \log 2$$

$$< 2^{k+1} \log 2 .$$

Given $x > 1$, $2^{k-1} < x < 2^k$ for some k and

$$\varphi(x) \leq \varphi(2^k) < 2^{k+1} \log 2 < (4 \log 2) \cdot x . \qquad \square$$

We will also need the following consequence of Theorems 1 & 2:

Proposition $\displaystyle\int_1^\infty \frac{\varphi(x) - x}{x^2}\, dx$ converges!

Proof: Set $f(t) := \varphi(e^t) e^{-t} - 1$ (piecewise C^0 on $[0, \infty)$). By Theorem 2, $|\varphi(e^t) e^{-t}| < C e^t e^{-t} = C$ on $[0, \infty)$. So Theorem 1 applies, and we have that

$$\int_0^\infty f(t)\,dt \underset{\substack{x=e^t \\ \log x = t}}{=} \int_1^\infty \frac{\varphi(x)-x}{x}\cdot\frac{dx}{x}$$ converges provided that the Laplace transform

$g(z) = \int_0^\infty f(t)e^{-zt}\,dt \in Hol(U_0)$ extends to $\tilde{g} \in Hol(\overline{U_0})$. For $Re(z) > 0$,

$$\begin{aligned} g(z) &= \int_0^\infty \frac{\varphi(e^t)-e^t}{e^t} e^{-zt}\,dt \\ &= \int_0^\infty \frac{\varphi(e^t)-e^t}{e^{2t}} e^{-zt} \underbrace{e^t\,dt}_{dx} \\ &\underset{(x=e^t)}{=} \int_1^\infty \frac{\varphi(x)-x}{x^{z+2}}\,dx \\ &= \frac{\Phi(z+1)}{z+1} - \frac{1}{z} \end{aligned}$$

which extends to a meromorphic function on $Re(z) > -\frac{1}{2}$ with no poles on the line $Re(z)=0$. (This is Theorem 3 of Lecture 18, with $s=z+1$.) So our $\tilde{U}$ is just $\{Re(z) > -\frac{1}{2}\} \setminus \text{poles}$ $(\supset \overline{U_0})$, and we have met the conditions of Theorem 1. $\square$

$\boxed{\text{Theorem 3}}$ $\varphi(x) \sim x$, i.e. $\lim_{x\to\infty} \frac{\varphi(x)}{x} = 1$.

Proof: We must show $\limsup_{x\to\infty} \frac{\varphi(x)}{x} \le 1$ and $\liminf_{x\to\infty} \frac{\varphi(x)}{x} \ge 1$. If the first fails, $\exists\, \delta > 0$ s.t. for certain arbitrarily large numbers y, $\varphi(y) > (1+2\delta)y$, hence $\varphi(y) > (1+2\delta)y > (1+\delta)x$ for $y < x < \rho y$, where $\rho := \frac{1+2\delta}{1+\delta}$. But then

$$\int_y^{\rho y} \frac{\varphi(x)-x}{x^2}\,dx > \int_y^{\rho y} \frac{\delta}{x}\,dx = \delta \log(\rho)$$

$$\big(\varphi(x) \ge \varphi(y) > (1+\delta)x\big)$$

for those same numbers y, contradicting the existence of the integral in the Proposition.

[**Exercise:** Show that we cannot have $\liminf_{x\to\infty} \frac{\varphi(x)}{x} < 1$ by the same approach, by considering intervals $\theta y < x < y$ with $\theta < 1$, where $\varphi(x) < (1-\delta)x$.]

This completes the proof. $\square$

Now let

$\pi(x) :=$ number of primes $\le x$

Prime Number Theorem $\pi(x) \sim \dfrac{x}{\log(x)}$

Proof: We have

$$\varphi(x) = \sum_{\substack{p \le x \\ p \text{ prime}}} \log p \le \sum_{\substack{p \le x \\ p \text{ prime}}} \log x = \pi(x) \log x$$

$$\Longrightarrow \quad \pi(x) \cdot \frac{\log x}{x} \ge \frac{\varphi(x)}{x}. \qquad (*)$$

Further, for $1 < y < x$,

$$\pi(x) = \pi(y) + \sum_{\substack{y < p \le x \\ p \text{ prime}}} 1 \le \pi(y) + \sum_{\substack{y < p \le x \\ p \text{ prime}}} \frac{\log p}{\log y}$$

$$\Longrightarrow \quad \pi(x) < y + \frac{\varphi(x)}{\log y}. \qquad (**)$$

Taking $y = \dfrac{x}{\log^2 x}$ in $(**)$,

$$\pi(x) \frac{\log x}{x} < \left(\frac{x}{\log^2 x} + \frac{\varphi(x)}{\log x - 2\log(\log x)} \right) \frac{\log x}{x}$$

$$= \frac{1}{\log x} + \frac{\varphi(x)}{x} \cdot \frac{\log x}{\log x - 2\log(\log x)}$$

which together with $(*)$ gives

$$\frac{\varphi(x)}{x} \le \pi(x)\frac{\log x}{x} \le \frac{1}{\log x} + \frac{\varphi(x)}{x}\,\frac{\log x}{\log x - 2\log(\log x)}.$$

$x \to \infty$

(squeeze lemma)

$$1$$

Whereupon Theorem 3 applies to give the result. $\square$

<u>Remark</u>// The function

$$\mathrm{Li}(x) := \int_0^x \frac{dt}{\log t}$$

is actually known to do a better job at approximating $\pi(x)$ than $\dfrac{x}{\log(x)}$. By a result of Schoenfield, <u>if</u> the Riemann hypothesis holds then one has

$$|\pi(x) - \mathrm{Li}(x)| \le \frac{1}{8\pi}\sqrt{x}\log x \quad \text{for all } x \ge 2657. \text{ //}$$