Lecture 2: Riemann Mapping Theorem

Here are three key results to bear in mind while reading these notes; $U$ denotes a region.

Montel: $f \in \text{Hol}(U)$ implies $\{f_n\}$ uniformly bounded

Normal: for any sequence $\{f_n\} \in \text{Hol}$, $\exists$ subsequence $\{f_{n_k}\}$ uniformly convergent on all compact subsets of $U$.

Hermite: Given $\{f_n\} \subset \text{Hol}(U)$ normally converging to $f$,
with each $f_n$ nowhere zero on $U$, either

* $f$ is nowhere zero on $U$

OR

* $f$ is identically zero on $U$.

Schwarz: Given $f \in \text{Hol}(D_1)$, $f(D_1) \subset D_1$, $f(0) = 0$.
Then $|f'(0)| \leq 1$, and if $|f'(0)| = 1$ then $f$ is a rotation ($= f(0) \cdot z$). These are the only conformal automorphisms fixing 0, and so one can say (with the same assumptions on $f$)

$$|f'(0)| = 1 \iff f \in \text{Aut}(D_1).$$
I. The statement

Throughout this lecture, \( \mathcal{R} \) denotes a simply connected region \( \subset \mathbb{C} \) which is not all of \( \mathbb{C} \).

\[ \text{RMT} \quad \mathcal{R} \text{ is biholomorphic to } D_1. \]

\[ \text{Corollary} \quad \text{Given } z_0 \in \mathcal{R}, \text{ there exists a unique function } f \in \text{hol}(\mathcal{R}) \text{ such that} \]

- \( f(z_0) = 0 \)
- \( f'(z_0) \in \mathbb{R}_+ \)
- \( f \) is 1-to-1
- \( f \) is "onto" the unit disk: \( f(\mathcal{R}) = D_1. \)

Proof of Cor. (assuming RMT):

Existence follows from RMT, and composing with any of the maps \( \phi_\alpha(z) := \frac{2 - \alpha}{1 - \alpha z} \) to translate \( f(z_0) \) to 0, then with a rotation to make \( f(z_0) \) positive real.

\( \dagger \) will also write \( \phi_\alpha(z) := \frac{z - 2}{1 - 2z} \).
For uniqueness, suppose \( f \) and \( g \) are two such functions. Then \( f \circ g^{-1} \) is a holomorphic automorphism of \( D_1 \) hence must be of the form \( e^{i\theta} \frac{z - \zeta}{1 - \overline{\zeta} z} \). Now, \( f, g : z \to 0 \Rightarrow (f \circ g^{-1})(0) = 0 \Rightarrow \zeta = 0 \Rightarrow (f \circ g^{-1})(z) = e^{i\theta} z \).

But \( (f \circ g^{-1})'(0) = f'(g^{-1}(0))g'(0) = f'(0)/g'(0) > 0 \Rightarrow e^{i\theta} = 1 \).

So \( (f \circ g^{-1})(z) = z \), i.e. \( f \circ g^{-1} = \text{id}_{D_1} \Rightarrow f = g \).

Why not \( C \subseteq D_1 \)? (Certainly \( z \to \frac{z}{1 + |z|} \) shows that \( C \sim D_1 \).) Answer: Liouville.

The fact that Schwarz enters above is interesting, because the idea of the proof of RMT itself comes from the Schwarz Lemma: for \( f : D_1 \to D_1 \) holo. with \( f(0) = 0 \),

\[
\text{if } f \text{ is biholo., } f \text{ is biholo. } \iff \text{if } f' \text{ is as large as possible.}
\]
Given \( z_0 \in \mathbb{D} \), consider holomorphic functions \( f : \mathbb{D} \to \mathbb{D} \) such that \( f(z_0) = 0 \) and \( |f'(z_0)| \) is "maximal".

Maybe this gives our biholomorphism!? But two questions immediately arise:

- Is the set of possible \( |f'(z_0)| \) even bounded?
  
  \[
  |f'(z_0)| = \frac{1}{2\pi} \left| \int_{\partial D} \frac{f(z)}{(z-z_0)^2} \, dz \right| \leq \frac{2\pi \|f\|_{L^2}}{2\pi} \leq \frac{1}{r}.
  \]

- Is the least upper bound obtained by some function, or is the set of possible values \( |f'(z_0)| \) "not closed at the top"?
II. The first proof

Write $\mathcal{H}(U,V) := \{ f \in \mathcal{H}(U) \mid f(U) \subset V \}$.

Lemma 1: Given $P \in U \subset \mathbb{C}$ open, $\overline{f}$ \text{ or } $\overline{\mathcal{H}(U,D,P)}$ nonempty family of functions all sending $P \to 0$. Then there exists an $f_0 \in \mathcal{H}(U,D,P)$ which is the normal limit of $\{ f_j \} \subset \overline{f}$, and which satisfies

$$|f_0'(P)| \geq |f'(P)| \quad (\forall f \in \overline{f}).$$

Proof: Set $\lambda := \sup \{ |f'(P)| \mid f \in \overline{f} \}$, which exists by the bracketed argument on the last page. By the definition of $\sup / \inf$, $\exists \{ f_j \} \subset \overline{f}$ with $|f_j'(P)| \to \lambda$. But $\{ f_j \}$ bounded by $1 \Rightarrow$ Menger compactness $\exists \{ f_{j_k} \}$ converging normally, hence to $f_0 \in \mathcal{H}(U)$.

Now

$$|f_{j_k} '(P) - f_0'(P)| = \frac{1}{2\pi} \left| \int_{\partial D(P,r)} \frac{f_{j_k}(z) - f_0(z)}{(z-P)^2} \, dz \right|$$

$$\leq \frac{1}{r} \| f_{j_k} - f_0 \|_{\partial D(P,r)} \quad (k \to \infty) \downarrow 0$$

compact
Hence \( |f_0'(p)| \leq \lambda \). As \( f_0 \) is a limit of functions \( f_0 \) in \( \mathcal{H}(U, D_1) \), \( f_0(U) \subset \overline{D_1} \). If \( \exists Q \in U \) with \( |f_0(Q)| = 1 \) then \( \text{MMP} \Rightarrow f_0 = e^{i\theta} \) (constant of modulus 1). This contradicts \( f_0(p) = 0 \), and so we conclude that \( f_0(U) \subset D_1 \).

Now let \( \Omega \) be as above, and set
\[
\Omega_f := \{ f \in \mathcal{H}(\Omega, D_1) \mid f \text{-to-} 1, f(p) = 0 \}.
\]

**Lemma 2:** \( \Omega_f \neq \emptyset \).

**Proof:**
- \( \Omega \subset C \setminus \{x \} \Rightarrow \overline{\Omega}(x) := x - d \) is nowhere zero
- \( \Omega \) simply conn. \( \Rightarrow \exists H \in \mathcal{H}(\Omega) \) with \( H^2 = J \)
- \( J \text{-to-} 1 \Rightarrow H \text{-to-} 1 \), AND
- if \( w \in \text{im}(H) \) then \( -w \in \text{im}(H) \)
- \( H \) open, with \( \Omega = D(\beta, r) \subset H(\Omega) \) and \( (-B) \cap H(\Omega) = \emptyset \)
- \( D(-\beta, r) \)
- \( \Rightarrow \) inversion in \( (-B) \) maps the image of \( H \) into \( (-B) \) which can then be translated & dilated into \( D_1 \).
More explicitly, \( f(x) + \beta \) has image outside the right disk \( D_r \), so \( \frac{1}{2} f(x) + \beta \) has image outside \( \overline{D_1} \), and

\[
f(x) := \frac{r}{2f(x) + \beta}
\]

maps \( \mathbb{R} \) into \( D_1 \). This is 1-to-1 (because composition of 1-1 with FLT) and composing \( f \) with \( f(x) \) (to send \( f(x) \) to 0) ensures that \( F := f^* \circ f \) sends \( P \to 0 \). So \( F \in \mathcal{F}^1 \). \( \square \)

**FIRST Proof of RMT:** It will suffice to show

(a) \( \mathcal{F} \neq \emptyset \)

(b) \( \exists f_0 \in \mathcal{F} \) s.t. \( |f_0'(p)| = \sup_{h \in \mathcal{F}} |h'(p)| \)

(c) if \( g \in \mathcal{F} \) has \( |g'(p)| = \sup_{h \in \mathcal{F}} |h'(p)| \), then \( g(D) = D_1 \).

(a) done (Lemma 2)

(b) We only need to check that the "\( f_0 \)" produced by Lemma 1 is 1-1.

Let \( \beta \in \mathbb{R} \) and look at

\[
g_j(z) := f_j(z) - f_j(\beta) \in \text{dial } (D_1 \setminus \{\beta\}) \]

\( f_j \) 1-1 \( \Rightarrow g_j \) is nonzero 0. Now Hurwitz
\[ \Rightarrow \text{the normal limit of the } \{g_j\} \text{ (namely, } f_0(x)-f_0(y)) \]
is either nowhere \(0\) or identically \(0\). Suppose the latter: \(f_0(x) \equiv f_0(y) \text{ (constant)} \Rightarrow \)
\[ 0 = \left| f_0'(P) \right| = \sup \{ |h'(P)| \mid h \in \mathcal{F} \}, \]
\text{Lemma 1} \text{ Lemma 2}

which means that \(h'(P) = 0 \) \((\forall h \in \mathcal{F} \neq \emptyset)\),
contradicting that each \(h\) is \(1-1\).
Therefore \(f_0(x) - f_0(y)\) must be nowhere \(0\)
on \(\Omega \setminus \{\beta\}\), meaning \(f_0(x) \neq f_0(y)\) for \(x \neq y\).
Since \(\beta\) was arbitrary, \(f_0\) is \(1\)-to-\(1\).

\((c)\) Take \(g \in \mathcal{F}\) with maximal \(\|g'(P)\|\).

Let \(Q \in D_1\) be such that \(Q \notin g(Q)\).
(We are after a contradiction \(\Rightarrow \text{i.e.}
\text{to construct some } P \in \mathcal{F} \text{ with bigger } \|g'(P)\|\).)
Set \(\phi(P) := \frac{g(z) - Q}{1 - Q \cdot g(z)} \text{. This is still } 1-1,\)
with \(\phi(Q) \in D_1\), and nowhere vanishing to boot.

Together with the fact that \(\Omega\) is simply conn.,
this ensures the existence of \(\Psi \in \text{hol}(\Omega)\)
with \(\Psi^2 = \emptyset\). Obviously \(\Psi \notin \mathcal{F}\) \(\text{(it's still nowhere)}\),
so \(p-1\)

\[ p(x) := \frac{\psi(x) - \psi(z)}{1 - \psi(z) \psi(x)} , \]

which does belong to \( \mathcal{G} \).

Actually, let's "rephrase" this construction in terms of

\[
\phi_{\psi}(x) = \frac{x - \psi}{1 - \psi x} , \quad \phi_{\psi}(z) = \frac{x - \psi(z)}{1 - \psi(z) x} , \quad \text{and} \quad S(x) = x^2 .
\]

\[
p = \phi_{\psi}(p) \circ \psi \implies \phi_{\psi}(p) \circ p = \psi \implies \\
S \circ \phi_{\psi}(p) \circ p = S \circ \psi = \psi^2 = \psi = \phi_{\psi} \circ g .
\]

So, \( g = \phi_{\psi}^{-1} \circ S \circ \phi_{\psi}(p) \circ p = : h \circ p \), where \( h : D_1 \to D_1 \) is not an automorphism (because \( S \)), but takes \( \Omega \mapsto \psi(p) \mapsto \psi(p)^2 = \psi(p) + Q \mapsto 0 \). Not 1-1

\[
\phi_{\psi}(p) \quad S \quad \phi_{\psi}^{-1}
\]

Hence, \( \text{Schwarz} \implies |h(0)| < 1 \implies \\
|g'(p)| = |h'(0)| \cdot |\rho'(p)| < |\rho'(p)| ,
\]

contradicting maximality of \( |g'(p)| \).
Appendix:

For reference, here is a proof of

**Hurwitz's Theorem**

Let \( f_n \in \mathcal{H}(U) \) with \( f_n \to f \)
uniformly on compact subsets. If \( (\forall a \in \mathbb{N}) \) \( f_n \) is
never 0 on \( U \), then either \( f \equiv 0 \) or \( f \) is never
0 on \( U \).

**Proof**: Assume \( f \neq 0 \). Given any \( z_0 \in U \),

\( \exists \, \varepsilon > 0 \) s.t. \( f(z) \neq 0 \) for \( z \in \overline{B^*(z_0, \varepsilon)} \subset U \).

(This is because zeroes of holomorphic functions are isolated.)

Set \( r := \min_{t \in \partial D(z_0, \varepsilon)} |f(t)| \); then for \( n \gg 0 \)

\[
\left\| \frac{1}{f_n} - \frac{1}{f} \right\|_{\partial D(z_0, \varepsilon)} = \left\| \frac{f - f_n}{f \cdot f_n} \right\|_{\partial D(z_0, \varepsilon)} \leq \frac{\left\| f - f_n \right\|_{\partial D(z_0, \varepsilon)}}{r} \xrightarrow{n \to \infty} 0.
\]

But we also know that \( f_n \to f \) uniformly on compact sets,
so \( \left\| f_n - f \right\|_{\partial D \to 0} \) and \( \left\| \frac{f_n'}{f_n} - \frac{f'}{f} \right\|_{\partial D \to 0} \), and

\[ 2\pi i \mathcal{N}(f, \partial D) = \oint_{\partial D} \frac{f'}{f} \, dz = \lim_{n \to \infty} \oint_{\partial D} \frac{f_n'}{f_n} \, dz = \lim_{n \to \infty} 2\pi i \mathcal{N}(f_n, \partial D) = 0. \]

So \( f \) has no zero at \( z_0 \). \( \square \)