Lecture 2: Riemann Mapping Theorem

Here are three key results to bear in mind while reading these notes; U denotes a region.

Montel: $\left.\begin{array}{l}\mathcal{F} \subset Hol(U) \\ \text{uniformly bounded}\end{array}\right\} \Rightarrow \mathcal{F}$ normal: for any sequence $\{f_n\} \subset \mathcal{F}$, $\exists$ subsequence $\{f_{n_k}\}$ uniformly convergent on all compact subsets of U.

Hurwitz: Given $\{f_n\} \subset Hol(U)$ normally converging to f, with each f_n nowhere zero on U, either

- f is nowhere zero on U

OR

- f is identically zero on U.

Schwarz: Given $f \in Hol(D_1)$, $f(D_1) \subset D_1$, $f(0) = 0$.
Then $|f'(0)| \leq 1$, and if $|f'(0)| = 1$ then f is a rotation ($= f'(0) \cdot z$). These are the only conformal automorphisms fixing 0, and so one can say (with the above assumptions on f)

$$|f'(0)| = 1 \iff f \in Aut(D_1).$$

I. The statement

Throughout this lecture, Ω denotes a simply connected region $\subset \mathbb{C}$ which is not all of $\mathbb{C}$.

RMT Ω is biholomorphic to D_1.

Corollary Given $z_0 \in \Omega$, there exists a unique function $f \in \mathcal{H}ol(\Omega)$ such that

- $f(z_0) = 0$
- $f'(z_0) \in \mathbb{R}_+$
- f is 1-to-1
- f is "onto" the unit disk: $f(\Omega) = D_1$.

Proof of Cor. (assuming RMT):

Existence follows from RMT, and composing with one of the maps† $\phi_\alpha(z) := \dfrac{z-\alpha}{1-\bar{\alpha} z}$ to translate $f(z_0)$ to 0, then with a rotation to make $f'(z_0)$ positive real.

† We'll also write $\tilde{\phi}_\alpha(z) := \dfrac{\alpha - z}{1-\bar{\alpha} z}$.

For *uniqueness*, suppose f & g are two such functions. Then $f \circ g^{-1}$ is a holomorphic automorphism of D_1 hence must be of the form $e^{i\varphi} \cdot \frac{z - \zeta}{1 - \bar{\zeta} z}$. Now,

$$f, g : z_0 \mapsto 0 \implies (f \circ g^{-1})(0) = 0$$
$$\implies \zeta = 0$$
$$\implies (f \circ g^{-1})(z) = e^{i\varphi} z.$$

But $(f \circ g^{-1})'(0) = f'(g^{-1}(0)) / g'(g^{-1}(0)) = f'(z_0)/g'(z_0) > 0$

$$\implies e^{i\varphi} = 1.$$

So $(f \circ g^{-1})(z) = z$, i.e. $f \circ g^{-1} = id_{D_1} \implies f = g$. $\square$

Why not $\mathbb{C} \underset{\text{biholo.}}{\cong} D_1$? (Certainly $z \mapsto \frac{z}{1 + |z|}$ shows that $\mathbb{C} \xrightarrow[\text{homeo.}]{\cong} D_1$.) <u>Answer</u>: Liouville.

The fact that Schwarz enters above is interesting, because the idea of the proof of RMT itself comes from the Schwarz Lemma: for $f : D_1 \to D_1$ holo. with $f(0) = 0$,

$$\boxed{\begin{array}{l} f \text{ is bijective} \\ \text{(hence a conformal equivalence)} \end{array}} \iff \boxed{\begin{array}{l} |f'(0)| \text{ is as} \\ \text{large as possible.} \end{array}}$$

Given $z_0 \in \Omega$, consider holomorphic functions $f: \Omega \to D_1$ such that $f(z_0) = 0$ and $|f'(z_0)|$ is "maximal".

Maybe this gives our biholomorphism!? But two questions immediately arise:

- Is the set of possible $|f'(z_0)|$ even bounded?

  [Yes: for some r, $\bar{D}(z_0, r) \subset \Omega \Rightarrow$

  $$|f'(z_0)| = \frac{1}{2\pi}\left|\int_{\partial D} \frac{f(z)}{(z-z_0)^2}\,dz\right| \leq \frac{2\pi r}{2\pi}\,\frac{\|f\|_{\partial D}}{r^2} \leq \frac{1}{r}.$$]

- Is the least upper bound obtained by some function, or is the set of possible values $|f'(z_0)|$ "not closed at the top"?

II. The first proof

Write $\mathcal{H}ol(U,V) := \{f \in \mathcal{H}ol(U) \mid f(U) \subset V\}$.

<u>Lemma 1</u>: Given $P \in U \subset \mathbb{C}$ open, $\tilde{\mathcal{F}} \subset \mathcal{H}ol(U, D_1)$ nonempty family of functions all sending $P \mapsto 0$. Then there exists an $f_0 \in \mathcal{H}ol(U, D_1)$ which is the normal limit of $\{f_j\} \subset \tilde{\mathcal{F}}$, and which satisfies

$$|f_0'(P)| \geq |f'(P)| \qquad (\forall f \in \tilde{\mathcal{F}}).$$

<u>Proof</u>: Set $\lambda := \sup\{|f'(P)| \mid f \in \tilde{\mathcal{F}}\}$, which exists by the bracketed argument on the last page.

By the definition of sup/lub, $\exists \{f_j\} \subset \tilde{\mathcal{F}}$ with $|f_j'(P)| \to \lambda$. But $\{f_j\}$ bounded by 1 $\underset{\text{Montel}}{\Longrightarrow}$ $\exists \{f_{j_k}\}$ converging normally, hence to $f_0 \in \mathcal{H}ol(U)$.

Now
$$|f_{j_k}'(P) - f_0'(P)| = \frac{1}{2\pi}\left|\oint_{\partial D(P,r)} \frac{f_{j_k}(z) - f_0(z)}{(z-P)^2}\,dz\right|$$
$$\leq \frac{1}{r}\|f_{j_k} - f_0\|_{\underbrace{\partial D(P,r)}_{\text{compact}}}$$
$$\xrightarrow{(k\to\infty)} 0$$

Hence $|f_0'(P)| = \lambda$. As f_0 is a limit of functions in $\mathcal{H}ol(U, D_1)$, $f_0(U) \subset \overline{D_1}$. If $\exists\, Q \in U$ with $|f_0(Q)| = 1$ then MMP $\Rightarrow f_0 \equiv e^{i\theta}$ (constant of modulus 1). This contradicts $f_0(P) = 0$, and so we conclude that $f_0(U) \subset D_1$. $\square$

Now let Ω be as above, and set

$$\mathcal{F} := \{ f \in \mathcal{H}ol(\Omega, D_1) \mid f \text{ 1-to-1}, f(P) = 0 \}.$$

<u>Lemma 2</u>: $\mathcal{F} \neq \emptyset$.

Proof: • $\Omega \subset \mathbb{C} \setminus \{\alpha\} \Rightarrow J(z) := z - \alpha$ is nowhere zero

• Ω simply conn. $\Rightarrow \exists\, H \in \mathcal{H}ol(\Omega)$ with $H^2 = J$

• J 1-to-1 $\Rightarrow H$ 1-to-1, AND
if $w \in \mathrm{im}(H)$ then $-w \notin \mathrm{im}(H)$

$\Rightarrow H$ open, with $\mathcal{D} = D(\beta, r) \subset H(\Omega)$
and $\underset{D(-\beta, r)}{\underset{\shortparallel}{(-\mathcal{D})}} \cap H(\Omega) = \emptyset$

$\Rightarrow$ inversion in $(-\mathcal{D})$ maps the image of H into $(-\mathcal{D})$ which can then be translated & dilated into D_1.

More explicitly, $H(z)+\beta$ has image outside the r-disk D_r, so $\frac{2}{r}(H(z)+\beta)$ has image outside $\overline{D_1}$, and $f(z) := \dfrac{r}{2(H(z)+\beta)}$ maps Ω into D_1. This is 1-to-1 (because composition of 1-1 with FLT) and composing f with $\phi_{f(P)}$ (to send $f(P)$ to 0) ensures that $F := \phi_{f(P)} \circ f$ sends $P \mapsto 0$. So $F \in \mathcal{F}$. $\square$

FIRST PROOF of RMT: It will suffice to show

(a) $\mathcal{F} \neq \emptyset$

(b) $\exists f_0 \in \mathcal{F}$ s.t. $|f_0'(P)| = \sup_{h \in \mathcal{F}} |h'(P)|$

(c) if $g \in \mathcal{F}$ has $|g'(P)| = \sup_{h \in \mathcal{F}} |h'(P)|$, then $g(\Omega) = D_1$.

(a) done (Lemma 2)

(b) We only need to check that the "f_0" produced by Lemma 1 is 1-1.

Let $\beta \in \Omega$ and look at

$$g_j(z) := f_j(z) - f_j(\beta) \in Hol(\Omega \setminus \{\beta\});$$

f_j 1-1 $\Rightarrow g_j$ is nowhere 0. Now Hurwitz

$\Rightarrow$ the normal limit of the $\{g_j\}$ (namely, $f_0(z) - f_0(\beta)$) is either <u>nowhere 0</u> or <u>identically 0</u>. Suppose the latter: $f_0(z) \equiv f_0(\beta)$ (constant) $\Rightarrow$

$$0 = |f_0'(P)| \underset{\text{Lemma 1}}{=} \sup\left\{ |h'(P)| \;\middle|\; h \in \mathcal{F} \right\},$$

which means that $h'(P) = 0$ ($\forall\, h \in \mathcal{F} \overset{\text{Lemma 2}}{\neq} \emptyset$), contradicting that each h is 1-1.

Therefore $f_0(z) - f_0(\beta)$ must be nowhere 0 on $\Omega \setminus \{\beta\}$, meaning $f_0(z) \neq f_0(\beta)$ for $z \neq \beta$. Since β was arbitrary, f_0 is 1-to-1.

(c) Take $g \in \mathcal{F}$ with maximal $|g'(P)|$.

Let $Q \in D_1$ be such that $Q \notin g(\Omega)$.

(We are after a contradiction – i.e. to construct some $\rho \in \mathcal{F}$ with bigger $|\rho'(P)|$.)

Set $\phi(z) := \dfrac{g(z) - Q}{1 - \overline{Q} \cdot g(z)}$. This is still 1-1, with $\phi(\Omega) \subset D_1$, and nowhere vanishing to boot.

Together with the fact that Ω is simply conn., this ensures the existence of $\psi \in \mathrm{Hol}(\Omega)$ with $\psi^2 = \phi$. Obviously $\psi \notin \mathcal{F}$ (it's still nonvanishing), so put

$$\rho(z) := \frac{\psi(z) - \psi(P)}{1 - \overline{\psi(P)}\,\psi(z)},$$

which does belong to $\mathcal{F}$.

Actually, let's "rephrase" this construction in terms of

$$\phi_Q(z) = \frac{z - Q}{1 - \bar{Q} z}, \quad \phi_{\psi(P)}(z) = \frac{z - \psi(P)}{1 - \overline{\psi(P)}\, z}, \text{ and } S(z) = z^2:$$

$$\rho = \phi_{\psi(P)} \circ \psi \implies \phi_{\psi(P)}^{-1} \circ \rho = \psi \implies$$

$$S \circ \phi_{\psi(P)}^{-1} \circ \rho = S \circ \psi = \psi^2 = \phi = \phi_Q \circ g.$$

So $g = \phi_Q^{-1} \circ S \circ \phi_{\psi(P)}^{-1} \circ \rho =: h \circ \rho$, where

$h : D_1 \to D_1$ is <u>not</u> an automorphism (because of $\underset{\substack{\uparrow \\ \text{not 1-1}}}{S}$),

but takes $0 \xmapsto{\phi_{\psi(P)}^{-1}} \psi(P) \xmapsto{S} \psi(P)^2 = \phi(P) = -Q \xmapsto{\phi_Q^{-1}} 0.$

Hence Schwarz $\implies |h'(0)| < 1 \implies$

$$|g'(P)| = |h'(0)| \cdot |\rho'(P)| < |\rho'(P)|,$$

contradicting maximality of $|g'(P)|$. $\square$

Appendix:

For reference, here is a proof of

$\boxed{\text{Hurwitz's Theorem}}$ Let $f_n \in \mathrm{Hol}(U)$, with $f_n \to f$ uniformly on compact subsets. If $(\forall n \geq N)$ f_n is never 0 on U, then either $f \equiv 0$ or f is never 0 on U.

Proof: Assume $f \not\equiv 0$. Given any $z_0 \in U$,

$\exists$ $\epsilon > 0$ s.t. $f(z) \neq 0$ for $z \in \bar{D}^*(z_0, \epsilon) (\subset U)$.

(This is because zeroes of holomorphic functions are isolated.)

Set $\mu := \min_{z \in \partial D(z_0, \epsilon)} |f(z)|$; then for $n \gg 0$

$$\left\| \frac{1}{f_n} - \frac{1}{f} \right\|_{\partial D(z_0,\epsilon)} = \left\| \frac{f - f_n}{f_n f} \right\|_{\partial D} \leq \frac{\| f - f_n \|_{\partial D}}{\frac{1}{2}\mu^2} \xrightarrow[(n \to \infty)]{} 0 .$$

But we also know that $f_n' \to f'$ uniformly on compact sets, so $\| f_n' - f' \|_{\partial D} \longrightarrow 0$ and $\left\| \frac{f_n'}{f_n} - \frac{f'}{f} \right\|_{\partial D} \to 0$, and

$$\underbrace{2\pi i\, n(f, \partial D)}_{\left(\substack{\#\text{ of zeroes of} \\ f \text{ enclosed by } \partial D}\right)} = \oint_{\partial D} \frac{f'}{f} dz = \lim_{n \to \infty} \oint \frac{f_n'}{f_n} dz = \lim_{n \to \infty} 2\pi i\, n(f_n, \partial D)$$

$$= 0 .$$

So f has no zero at z_0. $\square$