Lecture 20: An application of the PNT

In what follows, we'll use the Prime Number Theorem to prove the irrationality of

$$\zeta(3) = \sum_{k\geq 1} \frac{1}{k^3},$$

established by R. Apéry in 1978. (Here I follow a simpler approach due to F. Beukers.)

Let $d_n := \operatorname{lcm}\{1, 2, \dots, n\}$, $s, r \in \mathbb{N}$ with $r \geq s$.

Lemma 1: (a) in case $r>s$:

$$\int_0^1\int_0^1 \frac{-\log(xy)}{1-xy} x^r y^s \, dx\, dy \in \frac{1}{d_r^3}\mathbb{Z}$$

(b) in case $r=s$:

$$\int_0^1\int_0^1 \frac{-\log(xy)}{1-xy} x^r y^s \, dx\, dy = 2\Big\{\zeta(3) - \underbrace{\sum_{k=1}^{r} \frac{1}{k^3}}_{\text{also belongs to}}\Big\}$$

Proof: We claim, for $\sigma \geq 0$ and $r > s$, that

$$(*) \quad \int_0^1\int_0^1 \frac{x^{r+\sigma}y^{s+\sigma}}{1-xy}\, dx\, dy = \frac{1}{r-s}\left\{\frac{1}{s+\sigma+1} + \cdots + \frac{1}{r+\sigma}\right\}.$$

Indeed,

$$\left.\begin{aligned} \mathrm{LHS}(*) &= \sum_{k\geq 0} \int_0^1\int_0^1 x^{k+r+\sigma} y^{k+s+\sigma}\, dx\, dy \\ &= \sum_{k\geq 0} \frac{1}{(k+r+\sigma+1)(k+s+\sigma+1)} \end{aligned}\right\} (**)$$

$$= \sum_{k\ge 0} \frac{1}{r-s}\left\{\frac{1}{k+s+\sigma+1} - \frac{1}{k+r+\sigma+1}\right\}$$

$$= \text{RHS } (*),$$

where the switching of Σ & $\iint$ is easily justified on the basis of positivity of the $x^{k+r+\sigma} y^{k+s+\sigma}$ (absolute convergence argument).

Differentiating both sides of $(*)$ w.r.t. σ gives

$$\int_0^1\int_0^1 \frac{\log(xy)}{1-xy}\, x^{r+\sigma} y^{s+\sigma}\, dx\, dy = \frac{-1}{r-s}\left\{\frac{1}{(s+\sigma+1)^2} + \cdots + \frac{1}{(r+\sigma+1)^2}\right\},$$

and setting $\sigma = 0$ we have

$$\int_0^1\int_0^1 \frac{\log(xy)}{1-xy}\, x^r y^s\, dx\, dy = \frac{-1}{r-s}\left\{\frac{1}{(s+1)^2} + \cdots + \frac{1}{(r+1)^2}\right\}$$

$$= \frac{\text{integer}}{(r-s)\cdot(\operatorname{lcm}\{s+1,\dots,r\})^2},$$

in which the denominator divides d_r^3. This gives (a).

Taking $r = s$ in $(**)$ gives

$$\int_0^1\int_0^1 \frac{x^{r+\sigma} y^{r+\sigma}}{1-xy}\, dx\, dy = \sum_{k\ge 0} \frac{1}{(k+r+\sigma+1)^2},$$

and differentiating w.r.t. σ then setting $\sigma = 0$ $\Longrightarrow$

$$\int_0^1\int_0^1 \frac{\log(xy)}{1-xy}\, x^r y^r\, dx\, dy = \sum_{k\ge 0} \frac{-2}{(k+r+1)^3}$$

$$= -2\left\{\sum_{l\ge 1}\frac{1}{l^3} - \sum_{l=1}^{r}\frac{1}{l^3}\right\}. \quad \square$$

Remark // This clearly works for $r=0$, interpreting "$\sum_{l=1}^{0} \frac{1}{l^3}$" as 0. //

Lemma 2: For n sufficiently large, $d_n < 3^n$.

Proof:
$$d_n = \prod_{\substack{p \le n \\ (p \text{ prime})}} p^{\lfloor \log_p n \rfloor} < \prod_{\substack{p \le n \\ (p \text{ prime})}} p^{\log_p n} = n^{\pi(n)}.$$

(lcm$\{1,\dots,n\}$ is the product of prime powers $\le n$. The highest power of p which is $\le n$, is $\lfloor \log_p n \rfloor$.)

(that is, $\underline{\underline{n}}$. So this is just $\pi(n)$ copies of n multiplied together!)

Let $\epsilon > 0$ be such that $e^{1+\epsilon} < 3$. By the PNT,

$$\exists N \text{ s.t. } n \ge N \implies \pi(n) < (1+\epsilon)\frac{n}{\log n}$$
$$\implies \pi(n)\log n < (1+\epsilon) n$$
$$\overset{\exp}{\implies} n^{\pi(n)} < (e^{1+\epsilon})^n < 3^n. \quad \square$$

In the final proof, we'll use (shifted) Legendre polynomials
$$P_n(x) := \frac{1}{n!}\left(\frac{d}{dx}\right)^n x^n(1-x)^n \in \mathbb{Z}[x].$$

The first few of these are
$$P_0 = 1, \quad P_1 = 1-2x, \quad P_2 = 6x^2-6x+1,$$

$$P_3 = 20x^3 - 30x^2 + 12x - 1,$$
$$P_4 = 70x^4 - 140x^3 + 90x^2 - 20x + 1, \quad \text{etc.}$$

$\boxed{\text{Theorem}}$ $\zeta(3) \notin \mathbb{Q}$.

Proof: Consider

$$(\#) \qquad \int_0^1 \int_0^1 \frac{-\log(xy)}{1-xy} \underbrace{P_n(x) P_n(y)}_{\text{terms of form } x^r y^s,\ r,s \le n} \, dx\, dy .$$

Lemma 1 $\Rightarrow$ $(\#) = (A_n + \zeta(3) B_n)/d_n^3$, for some $A_n, B_n \in \mathbb{Z}$. Now

$$\int_0^1 \frac{dz}{1-(1-xy)z} = \left. \frac{-\log(1-(1-xy)z)}{1-xy} \right|_{z=0}^{1}$$
$$= \frac{-\log(xy)}{1-xy},$$

so
$$(\#) = \int_0^1 \int_0^1 \int_0^1 \frac{P_n(x)\, P_n(y)}{1-(1-xy)z} \, dx\, dy\, dz .$$

Next we perform integration by parts in the inner integral:
$$\int_0^1 \frac{\frac{1}{n!}\left(\frac{d}{dx}\right)^n x^n(1-x)^n}{1-(1-xy)z} \, dx = \int_0^1 \frac{\left[\frac{1}{n!}\left(\frac{d}{dx}\right)^{n-1} x^n (1-x)^n\right](yz)}{(1-(1-xy)z)^2} \, dx$$

$u = 1/(1-(1-xy)z)$, $dv = \frac{1}{n!}\left(\frac{d}{dx}\right)^n x^n (1-x)^n dx$

$du = \frac{-yz\, dx}{(1-(1-xy)z)^2}$, $v = \frac{1}{n!}\left(\frac{d}{dx}\right)^{n-1} x^n (1-x)^n$

vanishes at 0, 1

Continuing the process, the successive du's spew out a

$$n!\,(yz)^n$$

which leaves us in the end with

$$\int_0^1 \frac{\cancel{\frac{1}{n!}}\, x^n(1-x)^n\,(yz)^n\,\cancel{n!}}{(1-(1-xy)z)^{n+1}}\,dx\,.$$

Thus

$$(\#) = \int_0^1\int_0^1\int_0^1 \frac{(xyz)^n(1-x)^n P_n(y)}{(1-(1-xy)z)^{n+1}}\,dx\,dy\,dz$$

dz/dw

$$= \int_0^1\int_0^1\int_0^1 \frac{(xy)^n(1-w)^n}{(1-(1-xy)w)^n}\;\frac{(1-x)^n P_n(y)\left(\frac{-xy}{(1-(1-xy)w)^2}\right)}{\left(1-\frac{(1-xy)(1-w)}{1-(1-xy)w}\right)^{n+1}}\,dx\,dy\,dw$$

$$\left(1-\frac{(1-xy)(1-w)}{1-(1-xy)w}\right) = \frac{xy}{1-(1-xy)w}$$

change of variable $w \leftrightarrow z$:
$w = \frac{1-z}{1-(1-xy)z}$ / $z = \frac{1-w}{1-(1-xy)w}$

$\Longrightarrow\ dz = \frac{dz}{dw}\,dw = \frac{-xy\,dw}{(1-(1-xy)w)^2}$

$$= \iiint \frac{(1-w)^n(1-x)^n P_n(y)}{1-(1-xy)w}\,dx\,dy\,dw$$

(##)

$$= \iiint \frac{(1-w)^n(1-x)^n\,y^n(1-y)^n\,(xw)^n}{(1-(1-xy)w)^{n+1}}\,dx\,dy\,dw\,.$$

Carry out the same n-fold $\int$-by-parts for y as done for x above. Recall this produced (besides $x^n(1-x)^n$) $n(yz)^n$.

Consider the function

$$\phi(x,y,w) = \frac{xyw(1-x)(1-y)(1-w)}{1-(1-xy)w}$$

on $[0,1]^{\times 3}$.

The partial derivatives must all vanish in order for ϕ to have a maximum. In order for them to vanish somewhere other than the boundary of $[0,1]^{\times 3}$, the "interesting" terms in $\partial_x\phi, \partial_y\phi, \partial_w\phi$ must $=0$. These are

$$\left.\begin{array}{l} 1-w-2x-x^2yw+2xw \\ 1-w-2y-xy^2w+2yw \\ 1-2w+w^2-xyw^2 \end{array}\right\} (\equiv 0).$$

The last one says $\frac{(w-1)^2}{w^2} = xy$. In the first of these, make the substitution $-x^2yw = -x\frac{(w-1)^2}{w}$, and multiply through by w. This leads to

$$x = \frac{w}{1+w} \quad \left(= y \text{ by a "symmetric" argument using the } 2^{\text{nd}} \text{ partial term}\right).$$

So the 3^{rd} partial term becomes

$$0 = 1-2w+w^2 - \frac{w^2}{(1+w)^2}w^2$$

$$\Rightarrow w = \frac{1}{\sqrt{2}} \Rightarrow x = \frac{1}{\sqrt{2}+1} = \sqrt{2}-1 \; (=y).$$

Plugging these back into ϕ gives

$$\phi\left(\sqrt{2}-1, \sqrt{2}-1, \tfrac{1}{\sqrt{2}}\right) = \ldots = (\sqrt{2}-1)^4,$$

which is $\therefore$ the maximum of ϕ on $[0,1]^3$.

So $(\#) = \iiint_{[0,1]^3} \frac{(\phi(x,y,w))^n}{1-(1-xy)w}\,dx\,dy\,dw$

$$\leq (\sqrt{2}-1)^{4n} \iiint_{[0,1]^3} \frac{dx\,dy\,dw}{1-(1-xy)w}$$

$$= (\sqrt{2}-1)^{4n} \int_0^1\int_0^1 \frac{-\log(xy)}{1-xy}\,dx\,dy$$

$$\left(\text{using } \int_0^1 \frac{dw}{1-(1-xy)w} = \frac{-\log(xy)}{1-xy}\right)$$

$$= (\sqrt{2}-1)^{4n} \cdot 2\zeta(3).$$

(Lemma 1(b) + Remark)

Now from the form (##), we know $(\#) \neq 0$. Hence

$$0 < \underbrace{|A_n + B_n\zeta(3)|\, d_n^{-3}}_{(\#)} < 2\zeta(3)(\sqrt{2}-1)^{4n}$$

$$\Rightarrow 0 < |A_n + B_n\zeta(3)| < 2\zeta(3)\, d_n^3 (\sqrt{2}-1)^{4n}$$

$$< 2\zeta(3)\left(\frac{27}{(\sqrt{2}+1)^4}\right)^n .$$

Lemma 2

≈ 32

Finally suppose $\zeta(3) \in \mathbb{Q}$, i.e.

(†) $\exists\ P, Q \in \mathbb{N}$ s.t. $\zeta(3) = \frac{P}{Q}$.

Then, multiplying by Q, and taking n sufficiently large, we have

$$0 < \left| QA_n + B_n P \right| < 25(3)Q\left(\frac{27}{32}\right)^n < 1 \ldots$$

which, of course, is impossible, as $QA_n + B_n P \in \mathbb{Z}$.

So $(\dagger)$ cannot hold. $\square$