Lecture 22: Abel's Theorem

I. Elliptic functions

Let a lattice $\Lambda \subset \mathbb{C}$ be given, and write $\tilde{F}$ for a choice of $\tilde{F}_\Lambda$.

Definition An <u>elliptic function</u> (with respect to Λ) is $f \in \mathrm{Mer}(\mathbb{C})$ satisfying

$$(*) \qquad f(z+\omega) = f(z) \quad \forall\, \omega \in \Lambda,\, z \in \mathbb{C}.$$

Remark// (i) We need only check $(*)$ for $\omega = \omega_1, \omega_2$ (bases)

(ii) $f \in \mathrm{Hol}(\mathbb{C})$ & elliptic $\Rightarrow$ f constant (Liouville). //

Let an elliptic f be given.

Theorem 1 <u>Assume that the poles of f do not lie on $\partial\tilde{F}$.</u>

<u>Then $\sum_{p \in \tilde{F}} \mathrm{Res}_p(f) = 0$.</u>

Proof: $2\pi i \sum_{p \in \tilde{F}} \mathrm{Res}_p(f) = \int_{\partial \tilde{F}} f\, dz = 0$ since by periodicity, integrals on opposite sides cancel.

$\square$

Corollary A nonconstant elliptic function f has at least 2 poles in σ_α (counted with multiplicity).

Proof: By Remark (ii), it must have one.

By Theorem 1, there must either be another pole to cancel its residue, or the "one" pole must be at least double so that it can be residue-free. $\square$

Theorem 2 Assume that the zeroes and poles of f in σ_α do not lie on $\partial\sigma_\alpha$; denote them by $\{a_j\}$ with orders $\{m_j\}$. Then

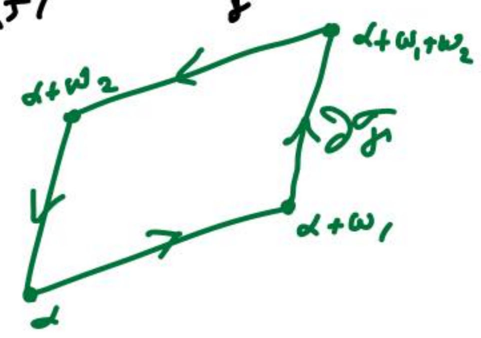
$$(**) \qquad \begin{cases} \sum_j m_j = 0 \\ \sum m_j a_j \underset{\Lambda}{\equiv} 0. \end{cases}$$

Proof: f elliptic $\Rightarrow f', f'/f$ elliptic

$$\Rightarrow 0 = \int_{\gamma_\alpha} \frac{f'}{f} dz = 2\pi i \sum_j \operatorname{Res}_{a_j}\left(\frac{f'}{f}\right) = 2\pi i \sum_j m_j.$$

Moreover, $\operatorname{Res}_{a_j}\left(z\frac{f'}{f}\right) = m_j a_j$, so

$$2\pi i \sum m_j a_j = \int_{\partial\sigma_\alpha} z\frac{f'}{f}\, dz$$



$$= \left(\int_{\alpha}^{\alpha+\omega_1} z \frac{f'}{f}\,dz - \int_{\alpha+\omega_2}^{\alpha+\omega_1+\omega_2} z \frac{f'}{f}\,dz \right)$$

$$+ \left(\int_{\alpha+\omega_1}^{\alpha+\omega_1+\omega_2} z \frac{f'}{f}\,dz - \int_{\alpha}^{\alpha+\omega_2} z \frac{f'}{f}\,dz \right)$$

$$= -\omega_2 \int_{\alpha}^{\alpha+\omega_1} \underbrace{\frac{f'}{f}\,dz}_{d\log f} + \omega_1 \int_{\alpha}^{\alpha+\omega_2} \underbrace{\frac{f'}{f}\,dz}_{d\log f}$$

$$= -\omega_2 \underbrace{2\pi i n} + \omega_1 \underbrace{2\pi i m}, \quad n, m \in \mathbb{Z}$$

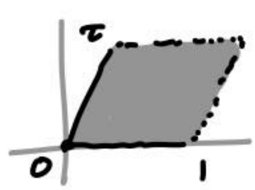
$$\in 2\pi i \Lambda.$$ (change in $\log f$ "along a period") $\square$

$\boxed{\text{II. Theta functions \& Abel's theorem}}$

Theorem 2 places considerable constraints on the locations and multiplicities of zeroes & poles of an elliptic function. It is of great interest to know whether, for a set of numbers a_j & multiplicities m_j satisfying (**), there exists f having exactly these a_j as its zeroes & poles (w/ mult. m_j).

For simplicity, we restrict to the setting

$\Lambda = \mathbb{Z}\langle 1, \tau \rangle$ $(\tau = t + is,\ s > 0)$, $\mathbb{C}/\Lambda =$



and introduce the (non-periodic hence non-elliptic) <u>theta function</u>

$$\Theta(z) := \sum_{n\in\mathbb{Z}} e^{\pi i\{n^2\tau + 2nz\}} \in \mathcal{H}ol(\mathbb{C}).$$

(To see that the sum is AC, take $N >> 0$ s.t. $Ns > 2|y|+1$. Then $\left|e^{\pi i\{n^2\tau+2nz\}}\right| = e^{-\pi n(ns+2y)} < e^{-\pi|n|}$ for $|n| > N$, and $\sum_{n\in\mathbb{Z}} e^{-\pi|n|}$ is obviously convergent.)

This has the properties

(a) $\underline{\Theta(-z) = \Theta(z)}$

(b) $\underline{\Theta(z+1) = \Theta(z)}$ [Exercise]

(c) $\underline{\Theta(z+\tau) = \Theta(z)\cdot e^{-2\pi i\left(\frac{\tau}{2}+z\right)}}$.

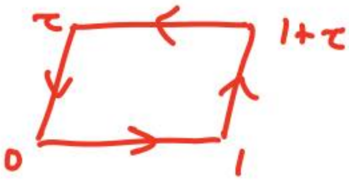
[to check (c), write $\Theta(\tau+z) = \sum_{n\in\mathbb{Z}} e^{\pi i\{n^2\tau+2nz+2n\tau\}}$

$= \sum_{n\in\mathbb{Z}} e^{\pi i\{(n+1)^2\tau+2(n+1)z-\tau-2z\}} \underset{\text{(reindex by } m=n+1)}{=} e^{-\pi i\tau-2\pi i z}\sum_{m\in\mathbb{Z}} e^{\pi i\{m^2\tau+2mz\}}$]

(d) Θ has a simple (order 1) zero at $\frac{\tau+1}{2}$ and nowhere else in $\overline{\mathcal{F}}$.

Proof: $\frac{1}{2\pi i}\int_{\partial\mathcal{F}} d\log\Theta \underset{(b)+(c)}{=} \frac{1}{2\pi i}\Big(\int_0^1 d\log\Theta - \overbrace{\int_\tau^{1+\tau} d\log\Theta}^{=\int_0^1 d\{-2\pi i(\frac{\tau}{2}+z)\} + d\log\Theta}\Big)$

$$= \frac{1}{2\pi i}\int_0^1 2\pi i\, dz = 1.$$



Since Θ is pole-free, it has one zero in $\tilde{\pi}$ with multiplicity 1. To locate the zero:

$$\frac{1}{2\pi i}\int_{\partial \tilde{\pi}} z \, d\log\theta =$$

$$\frac{1}{2\pi i}\left\{\int_1^{1+\tau} d\log\theta + \int_{1+\tau}^{\tau}\left[\tau \, d\log\theta + z(-2\pi i)\,dz + \tau\,(2\pi i)\,dz\right]\right\} =$$

$$\frac{1}{2\pi i}\left\{(-\pi i)\left[(\tau+2) + (2\times\text{integer})\right] + \left[\text{integer}\times(2\pi i\tau)\right] + (\pi i)(2\tau+1) - 2\pi i\tau\right\} =$$

$$\frac{1}{2}+\frac{\tau}{2}+m+n\tau, \qquad m,n\in\mathbb{Z} \text{ (which must be } 0\text{)}. \quad \square$$

Theorem 3 (Abel) Let $\{a_j\}$, $\{m_j\}$ satisfy $(**)$.

Then $\exists$ elliptic function with zeroes/poles $\{a_j\}$ & multiplicities $\{m_j\}$.

Proof: Consider

$$f(z) := \prod_j \Theta\left(z - a_j + \frac{\tau+1}{2}\right)^{m_j}.$$

Clearly $f(z+1) = f(z)$ by property (b). Also, using (c),

$$\frac{f(z+\tau)}{f(z)} = \prod_j \left(\frac{\Theta\left(\left\{z-a_j+\left(\frac{\tau+1}{2}\right)\right\}+\tau\right)}{\Theta\left(z-a_j+\frac{\tau+1}{2}\right)}\right)^{m_j}$$

$$= \prod_j \left(e^{-2\pi i\left(\tau+\frac{1}{2}+z-a_j\right)}\right)^{m_j}$$

$$= e^{-2\pi i\left(\tau+\frac{1}{2}+z\right)\cancelto{0}{\sum_j m_j}} \cdot e^{2\pi i \overbrace{\sum m_j a_j}^{=M+N\tau}}$$

$$= e^{2\pi i N\tau}.$$

The function $g(z) := e^{-2\pi i N z} f(z)$ will therefore satisfy $g(z+\tau) = g(z) = g(z+1)$. So it is Λ-periodic ($\Rightarrow$ elliptic), and the definition of f together with property (d) makes it clear that g has the desired zeroes & poles w/multiplicity. $\square$

Theorems 2 & 3 yield a bijection

$$\left\{\begin{array}{c}\text{meromorphic functions}\\ \text{(up to scale) on } \mathbb{C}/\Lambda\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{Configurations of points}\\ \text{with multiplicity on } \mathbb{C}/\Lambda\\ \text{satisfying (*\&*)}\end{array}\right\},$$

since any 2 functions with the same zeroes & poles (w/mult) have constant ratio (b/c holo-/periodic). These 2 theorems together are known as "Abel's theorem for genus 1" in algebraic geometry (and there is a corresponding, more complicated, result in higher genus).

<u>Remark on $\Gamma\backslash\mathfrak{h}$</u> // As we shall see, it is complex-analytically isomorphic to $\mathbb{C}$, which one can think

of as $\mathbb{P}^1 \setminus \{\infty\}$. More generally, if

$$\Gamma(N) := \ker\left\{\Gamma \to SL_2(\mathbb{Z}/N\mathbb{Z})\right\} \quad \left(\text{matrices } \underset{(N)}{\equiv} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$$

then $\Gamma(N)\backslash\mathfrak{h}$ is the complement of a finite set of points in a Riemann surface of genus

$$g_N = 1 + \frac{N^2(N-6)}{24} \prod_{\substack{p \mid N \\ p \text{ prime}}} \left(1 - \frac{1}{p^2}\right).$$

These RS's are called modular curves //

III. A canonical product

Recall that we've been taking $\Lambda = \mathbb{Z}\langle 1, \tau \rangle$, $\mathrm{Im}(\tau) > 0$.

Motivated by the theta function above, which has a simple zero at all Λ-translates of $\frac{\tau+1}{2}$, we adopt a simpler and more general point of view and construct the Weierstraß σ-function.

For this purpose let $\Lambda = \mathbb{Z}\langle \omega_1, \omega_2 \rangle$ $\left(\mathrm{Im}\left(\frac{\omega_2}{\omega_1}\right) > 0\right)$, and consider the genus-2 canonical product

$$\boxed{\sigma(z) := z \prod_{\omega \in \Lambda}{}' \underbrace{\left(1 - \frac{z}{\omega}\right) e^{\frac{z}{\omega} + \frac{1}{2}\left(\frac{z}{\omega}\right)^2}}_{\text{i.e. } E_2\left(\frac{z}{\omega}\right)},}$$

$\leftarrow$ omit $\omega = 0$

which converges to an <u>entire function</u> with <u>simple zeroes on Λ</u> (& nowhere else) due to the following

<u>Lemma</u>: $\sum_{\omega\in\Lambda}' \frac{1}{|\omega|^\lambda}$ converges for $\lambda > 2$.

Proof: Pick $d \in \mathbb{N}$ s.t. $\operatorname{diam}(\mathcal{F}_\Lambda) \le d$, and let

$$A_n := \{ z \in \mathbb{C} \mid |z| \in [n-1, n) \}$$
$$\subset \{ z \in \mathbb{C} \mid |z| \in [(n-1)-d, n+d) \} =: \mathcal{A}_n .$$

(Note that $\operatorname{area}(\mathcal{A}_n) \le C \cdot n$, C = constant depending on d.)

Set $k_n :=$ # of Λ-translates of $\mathcal{F}_\Lambda$ meeting A_n
$\ge |\Lambda \cap A_n|$.

Since all of the translates are contained in $\mathcal{A}_n$,

$$k_n \cdot \operatorname{area}(\mathcal{F}_\Lambda) \le \operatorname{area}(\mathcal{A}_n) \le C \cdot n$$

$$\Longrightarrow \quad |\Lambda \cap A_n| \le C' \cdot n \quad (C' = C/\operatorname{area}(\mathcal{F}_\Lambda))$$

But $\sum_{\omega\in\Lambda}' \frac{1}{|\omega|^\lambda} = \text{finite sum} + \underbrace{\sum_{n\ge N} \sum_{\omega \in \Lambda\cap A_n} \frac{1}{|\omega|^\lambda}}_{\le C' \cdot \sum_{n \ge N} \frac{n}{(n-1)^\lambda}}$ which converges for $\lambda > 2$. $\square$

Remark // With $\omega_1 = 1$, $\omega_2 = \tau$, $\Theta(z) = Ce^{h(z)}\sigma(z + \frac{\tau+1}{2})$,

where $h(z) = -\frac{\eta_1}{2}z^2 - \left(\frac{\eta_1\tau}{2} + \frac{\eta_2}{2} + \pi i\right)z$. [Exercise] //

(see next lecture)

The Lemma has another important consequence: the

$$\boxed{s_m(\Lambda) := \sum_{\omega\in\Lambda}{}' \frac{1}{\omega^m}}$$

converge, and we will have particular use for

$$\boxed{g_2(\Lambda) := 60\, s_4(\Lambda) \quad \text{and} \quad g_3(\Lambda) := 140\, s_6(\Lambda)}$$

later. (Note that the odd $\{s_m\}$ are zero.)