Lecture 23: Elliptic functions & elliptic curves

I. Weierstraß $\sigma^-$ & $\sigma^+$ functions

Let $\Lambda := \mathbb{Z} \langle \omega_1, \omega_2 \rangle \subset \mathbb{C}$, $\text{Im} \left( \frac{\omega_2}{\omega_1} \right) > 0$.

In Lecture 22 we proved that

$$\sigma(2) := 2 \prod_{\omega \in \Lambda, \omega \neq 0} \left(1 - \frac{2}{\omega} \right) e^{\frac{2}{\omega} + \frac{1}{2} \left( \frac{2}{\omega} \right)^2} \quad \text{defines an entire function with simple zeros on } \Lambda \text{ (only).}$$

Let $\omega_0 \in \Lambda$; then

$$\sigma(2 + \omega_0) = \frac{(2 + \omega_0) \prod_{\omega \in \Lambda} \left(1 - \frac{2 + \omega_0}{\omega} \right) e^{\frac{2 + \omega_0}{\omega} + \frac{1}{2} \left( \frac{2 + \omega_0}{\omega} \right)^2}}{\sigma(2) \prod_{\omega \in \Lambda} \left(1 - \frac{\omega_0}{\omega} \right) e^{\frac{\omega_0}{\omega} + \frac{1}{2} \left( \frac{\omega_0}{\omega} \right)^2}},$$

and logarithmically differentiating yields

$$\frac{d}{dz} \log \left( \frac{\sigma(2 + \omega_0)}{\sigma(2)} \right) = \frac{1}{2 + \omega_0} - \frac{1}{z + \omega_0} + \sum \left\{ \frac{1}{2 + \omega_0 - 2 - \omega + \omega_0} \right\},$$

almost, but not yet, a collapsing sum. Need to differentiate one more...

$$\left( \frac{d}{dz} \right)^2 \log \left( \frac{\sigma(2 + \omega_0)}{\sigma(2)} \right) = \frac{-1}{(2 + \omega_0)^2} + \frac{1}{z^2} + \sum \left\{ \frac{-1}{(2 + \omega_0 - \omega)} + \frac{1}{(z - \omega)} \right\},$$

can now collapse. Note that $\sum$ contains no $\frac{-1}{(2 + \omega_0)^2}$ or no $\frac{1}{z^2}$.

$$= \frac{-1}{(2 + \omega_0)^2} + \frac{1}{z^2} + \frac{-1}{z^2} + \frac{1}{(2 + \omega_0)^2}$$

$$= 0.$$
So \( \log \left( \frac{\sigma(\omega + \omega_0)}{\sigma(\omega)} \right) = \eta(\omega_0) \pm 2 \sigma(\omega_0) \) (is linear).

Now clearly

\[
\sigma(-\omega) = \sigma(\omega) \prod_{\omega} (1 + \omega^2) e^{-\frac{\omega^2}{2} - \text{Re} \left( \frac{\omega}{\omega} \right)}
\]

\[
= -\sigma(\omega) \quad \text{ (is odd)}
\]

\[
(\omega \to -\omega)
\]

From the above, we have

\[
\sigma(\omega + \omega_0) = \sigma(\omega) e^{\eta(\omega_0) \pm 2 \sigma(\omega_0)}
\]

and setting \( \omega = -\frac{\omega_0}{2} \) gives, assuming \( \frac{\omega_0}{\omega} \neq 1 \),

\[
\sigma\left(\frac{\omega_0}{2}\right) = \left[-\frac{\omega_0}{2}; \frac{\omega_0}{2}\right]
\]

\[
\sigma\left(\frac{\omega_0}{2}\right) e^{-\eta(\omega_0) \pm 2 \sigma(\omega_0)}
\]

\[
\Rightarrow \eta(\omega_0) = \frac{n_i}{2} + \sigma(\omega_0) \frac{\omega_0}{2}.
\]

This proves the 1st part of

**Theorem 1**

(a) \( \sigma(\omega + \omega_1) = -\sigma(\pm \omega) e^{\eta_1 \cdot (\omega + \omega_1)} \) \( (\omega = 1, 2) \)

\[\text{when } \eta_1 = \eta(\omega_1).\]

(b) \( \eta_1 \omega_2 - \eta_2 \omega_1 = 2n_i \) [Legendre relation]

**Proof of (b):**

Define the Weierstrass S-function

\[
S(\omega) = \frac{d}{d\omega} \log \sigma(\omega)
\]

\[
= \frac{1}{2} + \sum_{\omega \in A} \left( \frac{1}{\omega^2 + 1} + \frac{1}{\omega + 2} \right).
\]
Since $S$ has simple poles on $\Lambda$ and nowhere else, we have

$$2\pi i = 2\pi i \text{ Res}_0 S = 2\pi i \sum_{f \in \mathbb{C}} \text{ Res}_f S$$

$$= \int_{\gamma_0} S(z) \, dz = \int_{\gamma_0} \left( S(z) - S(z + \omega_1) \right) \, dz + \int_{\gamma_2} \left( S(z + \omega_2) - S(z) \right) \, dz$$

$$= -2\pi i \omega_1 + 2\pi i \omega_2.$$

**II. Weierstrass $\wp$-function**

Set $\wp(z) := -S'(z)$

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda} \left( \frac{1}{(z - \omega)^2} - \frac{1}{\omega^2} \right).$$

**Properties**

1. $\wp$ is even
   
   $$\text{Proof: } \, \text{if } \, S \text{ is odd, then } S'(z) \text{ is even.}$$

   $$\Rightarrow \, \wp'(z) = -\frac{2}{z^3} - \sum_{\omega \in \Lambda} \frac{1}{(z - \omega)^3} \text{ is odd.}$$

2. $\wp$ is elliptic
   
   $$\text{Proof: } \, \wp(t + \omega) = \wp(t) + \text{const} \quad \frac{1}{g_{2} \wp'(z) - g_{3}}$$

   $$\Rightarrow \, \wp'(z) \text{ is also elliptic.}$$

3. $\wp$ (resp. $\wp'$) is of mapping degree 2 (resp. 3) as a map from $\mathbb{C}/\Lambda$ to $\mathbb{P}^2$. 
\[ \text{Proof: let } x \to \infty, \text{ consider } P(x) - P(a). \text{ We want to show that it has 2 O's in } \mathbb{T}, \text{ counting multiplicity. It has one pole, with mult. } (-2), \text{ and } \sum \alpha_i = 0 \text{ (Lect. 22). Then implies the result. } P' \text{ is treated in the same way.} \]

4. \( P \) (resp. \( P' \)) has principal part \( \frac{1}{2} \) (resp. \( -\frac{1}{2} \)) at 0.

[Proof: obvious from property 1, \& differentiation.]

Though I haven't listed it, obviously \( P \) has double poles at each \( \omega_i, -\omega_i \) and no other poles. Also, just as with the \( \Theta \)-function, all elliptic functions can be expressed in terms of products of (powers of) translates of the \( \sigma \)-function, and one can write a formula for \( \Theta \) in this way: in fact,

\[ P(x) - P(a) = \frac{\sigma(x+a) \sigma(x-a)}{\sigma^2(x) \sigma(x)} \]  

[Exercise]

We now prove a different sort of "generation" result:

\[ \textbf{Theorem 2} \]

Let \( f \in \text{Mer}(\mathbb{C}/\Lambda) \).

Then \( f \) may be expressed as a rational function in \( P \) and \( P' \). \( \dagger \)

\[ \dagger \] Algebraist's version: \( \text{Mer}(\mathbb{C}/\Lambda) \cong \mathbb{C}(P, P') \cong \mathbb{C}[x, y]\) where \( x, y \) are indeterminates.
Proof: \( f \text{ elliptic} \Rightarrow \)

\[
f(z) = \frac{f(e^z + e^{-z})}{2} + \frac{f(e^z) - f(e^{-z})}{2} = f_e(z) + \frac{f_0(z)}{P'(z)}
\]

Even elliptic
Odd elliptic

So it suffices to treat even elliptic functions only, which we will show to be rational functions in \( \wp \) alone. Assume \( f \) even.

Lemmas: (a) \( \text{ord}_{\wp_0}(f) = m \Rightarrow \text{ord}_{-\wp_0}(f) = m \)
(b) if \( \wp_0 \not\equiv -\wp_0 \) then \( 2 \mid \text{ord}_{\wp_0}(f) \).

Proof: (a) For \( m > 0 \), \( f^{(k)}(-\wp_0) = (-1)^k f^{(k)}(\wp_0) \).
For \( m < 0 \), look at \( \frac{1}{f} \).
(b) \( \wp_0 \) is either \( 0, \wp_1, \wp_2, \) or \( \wp_1 + \wp_2 \) (the 4 2-torsion points of \( \mathcal{E}/\Lambda \)).
Assume \( m \geq 0 \) (otherwise use \( \frac{1}{f} \)).
If \( k \) is odd, \( f^{(k)}(\wp_0) \) is odd, and so
\(-f^{(k)}(\wp_0) = f^{(k)}(-\wp_0) = f^{(k)}(\wp_0)\)
\( \Rightarrow f^{(k)}(\wp_0) = 0 \Rightarrow \) leading term in power-series expansion is \( (z - \wp_0)^m \) even.

Continuing the proof, let \( u_i : (i \neq i, \ldots, r) \) be a family of points containing one representative from each class \( (u, -u) \mod \Lambda \) where \( f \) has no zero or pole, other than the class of \( \Lambda \) itself. Let

- \( m_i = \text{ord}_{u_i}(f) \), if \( 2u_i \not\equiv 0 \), and
- \( m_i = \frac{1}{2} \text{ord}_{u_i}(f) \), if \( 2u_i \equiv 0 \);
III. The associated elliptic curve

Since $C/L$ is 1-dimensional, one does not expect the transcendence degree of its field of meromorphic functions to exceed one. So the next step is to look for an algebraic relation between $\phi$ and $\phi'$. Write (in a neighborhood of 0)

- $\phi(x) = \frac{1}{x^2} + \sum_{\omega \in \Lambda} \left\{ \frac{1}{\omega^2} \left( 1 - \frac{x}{\omega} \right)^{-2} - \frac{1}{\omega^2} \right\}$

  $$= \frac{1}{x^2} + \sum_{\omega \in \Lambda} \left( \sum_{m=0}^{m+1} \frac{\omega^{m+1}}{\omega^2} \right) \left( \sum_{m=0}^{m+1} \frac{(-1)^m(-2)^m \omega^m}{m!} \right)^m$$

  $$= \frac{1}{x^2} + \sum_{m=1} \left\{ (m+1) s_{m+2}(\Lambda) \right\} x^m = \frac{1}{x^2} + 3s_2 x^2 + 5s_4 x^4 + \ldots,$$

- $\phi'(x) = -\frac{2}{x^3} + 6x y + 20x^3 + \ldots$

Now $(\phi')^2 = \frac{4}{x^2} - \frac{24s_2}{x^3} + \ldots + C - \ldots$
\[
4 \rho^3 = \frac{4}{\xi^6} + \frac{965\xi}{\xi^2} + \text{horo.} \quad \xi = 6056 + \ldots
\]
\[
(\rho')^2 - 4 \rho^3 = \frac{-6054 + \text{horo.}}{\xi^2} \quad \xi = -14056 + \ldots
\]
So
\[
(\rho')^2 - 4 \rho^3 + 6054 \rho = \text{horo.}
\]
Elliptic curve = C.
From the constant term alone, we clearly have
\[
C = -14056
\]
We have proved the

**Theorem 3**

\[
(\rho')^2 = 4 \rho^3 - g_2 \rho - g_3,
\]
where \(g_2 = 6054\) and \(g_3 = 14056\) depend on \(\Lambda\).

**Corollary**

The map \(\mathbb{C} \times \Lambda \to \{0\} \cup \{\infty\}\) parametrizes points on the nonsingular Weierstrass elliptic curve

\[
E_\Lambda := \{ (x, y) \mid y^2 = 4x^3 - g_2(\Lambda)x - g_3(\Lambda) \} \cup \{\infty\}
\]

actually gives an analytic isomorphism

\[
P : \mathbb{C}/\Lambda \leftrightarrow E_\Lambda
\]
(think of it as the Riemann surface of)

\[
\left(\frac{1}{\sqrt{4x^3 - g_2x - g_3}}\right)
\]

(Complex 1-form)
Proof: We will show $P$ is 1-1 and onto:

(onto) \[ E_{\Lambda} \text{ is connected and compact, as is } C/\Lambda \Rightarrow \text{map has closed image,} \]
so we are done by the open mapping theorem \( \Rightarrow \text{map has open image}. \)

(1-1) \[ P \text{ is the composition of } P \text{ with projection } E_{\Lambda} \rightarrow \mathbb{P}^1, \]
\( (x, y, z) \rightarrow a \).

Since the equation is quadratic in $y$,
the projection has mapping degree 2.
But mapping degrees multiply under composition, and the mapping degree of $P$ is 2.
Since $P'$ is odd,
the mapping degree + 1 is one.

More precisely,
\[ E_{\Lambda} = \{ [x: y: z] \in \mathbb{P}^2(\mathbb{C}) \mid 2y^2 = 4x^3 - 9xz^2 - 9z^3 \} \]
and \( \infty \) is \( [0: 1: 0] \). Set
\[ C_i := P(\frac{\omega_i}{z}) , \]
where \( \omega_3 = \omega_1 + \omega_2 \). We know that \( P(z) - P(\frac{\omega_i}{z}) \) has a zero of order 2 at \( \frac{\omega_i}{z} = \infty \); so \( P'(z) \) has a zero of order one there. But then
\[ \frac{(P'(z))^2}{(P(z) - e_1)(P(z) - e_2)(P(z) - e_3)} \]
is zero & pole-free \( \Rightarrow \) constant.
and we conclude that

\[ 4x^3 - g_2 x - g_3 = 4 \prod_{i=1}^3 (x - e_i). \]

Since \((F')^3\) has only double roots, no two \(e_i\) can coincide, and so the discriminant of \(\prod(x - e_i) = x^3 - \frac{g_2}{3} x - \frac{g_3}{3}\)
must be nonzero:

\[
0 \neq \prod_{i < j} (e_i - e_j)^2 = \begin{vmatrix}
1 & 0 & -\frac{g_2}{3} & -\frac{g_3}{3} & 0 \\
0 & 1 & 0 & -\frac{g_2}{3} & -\frac{g_3}{3} \\
3 & 0 & \frac{2}{3} & 0 & 0 \\
0 & 3 & 0 & -\frac{g_2}{3} & 0 \\
0 & 0 & 3 & 0 & -\frac{g_3}{3}
\end{vmatrix} = \frac{g_2^3 - 27g_3^2}{-4^3}
\]

\[ \Rightarrow g_2^3 - 27g_3^2 \neq 0. \] This will turn out to be an important quantity later.