Lecture 24: Elliptic addition theorems

I. The Abel map

Brief review: Recall that for a lattice $\Lambda \subset \mathbb{C}$

- $s_m(\Lambda) := \sum'_{\omega \in \Lambda} \frac{1}{\omega^m} \quad (m > 2)$

  Note: zero for m odd!

- $g_2(\Lambda) := 60\, s_4(\Lambda), \quad g_3(\Lambda) := 140\, s_6(\Lambda)$

- $\wp(u) := \frac{1}{u^2} + \sum'_{\omega \in \Lambda} \left(\frac{1}{(u-\omega)^2} - \frac{1}{\omega^2} \right)$ satisfies

  u = coordinate on $\mathbb{C}$

  $\wp \in \mathcal{M}_{\mathbb{C}}(\mathbb{C}/\Lambda)$

  $$(\wp')^2 = 4\wp^3 - g_2(\Lambda)\wp - g_3(\Lambda)$$

- $P : \mathbb{C}/\Lambda \xrightarrow[(\wp,\, \wp')]{\cong} E_\Lambda := \{(x,y) \mid y^2 = 4x^3 - g_2(\Lambda)x - g_3(\Lambda)\} \cup \{\infty\}$

Now consider the Abel map (or abelian integral)

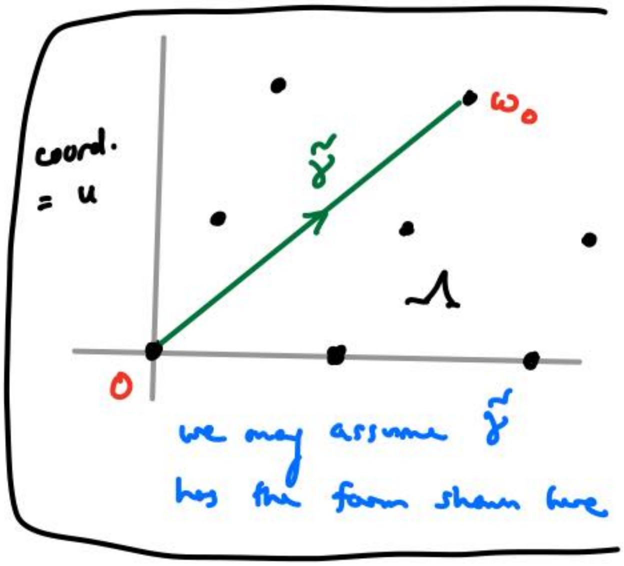
$$u : E_\Lambda \longrightarrow \mathbb{C}/\Lambda$$

$$\underset{(x_p, y_p)}{p} \longmapsto \int_\infty^p \frac{dx}{y} = \int_\infty^{x_p} \frac{dx}{\sqrt{4x^3 - g_2 x - g_3}}.$$

path on E_Λ

To see that u is well-defined, we must check that an integral around a loop $\gamma \in H_1(E, \mathbb{Z})$ gives a value in Λ. Since $P : \mathbb{C}/\Lambda \to E_\Lambda$ is an isomorphism, there is $\tilde{\gamma} \in H_1(\mathbb{C}/\Lambda, \mathbb{Z})$ with $P(\tilde{\gamma}) = \gamma$, and so coord. on $\mathbb{C}$

$$du = \frac{\wp'(u)\,du}{\wp'(u)} = \frac{d(\wp(u))}{\wp'(u)} = \wp^*\left(\frac{dx}{y}\right)$$



$\Rightarrow$

$$\int_\gamma \frac{dx}{y} = \int_{\wp_*(\tilde{\gamma})} dx/y \overset{\text{Stokes}}{=} \int_{\tilde{\gamma}} \wp^*\left(\frac{dx}{y}\right)$$

$$= \int_{\tilde{\gamma}} du = \omega_0 .$$

It is no harder to compute the composition of $\wp$ with $\mathcal{U}$:

$$\mathcal{U}(\wp(u_0)) = \int_\infty^{\wp(u_0)} \frac{dx}{y} = \int_{\wp(0)}^{\wp(u_0)} \frac{dx}{y} = \int_0^{u_0} \wp^*\left(\frac{dx}{y}\right) = \int_0^{u_0} du = u_0 .$$

$\boxed{\text{Theorem 1}}$ The Weierstraß $\wp$-function inverts the abelian integral above, in the precise sense that

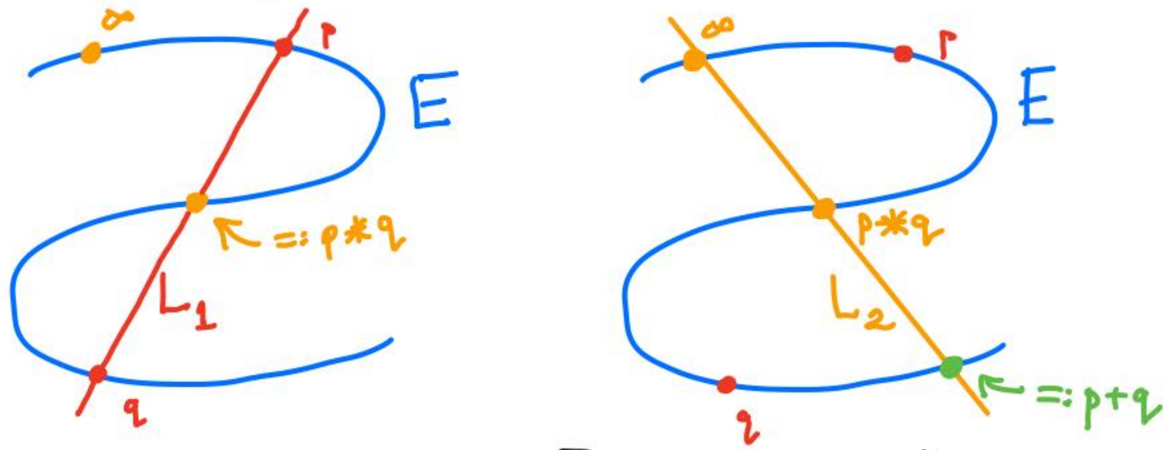
$$\int_\infty^{\wp(z)} \frac{dx}{\pm\sqrt{4x^3 - g_2 x - g_3}} = z \quad \text{and} \quad \wp\left(\int_\infty^{x_0} \frac{dx}{\sqrt{4x^3 - g_2 x - g_3}}\right) = x_0 .$$

(Alternatively, $\wp : \mathbb{C}/\Lambda \to E_\Lambda$ and $\mathcal{U}: E_\Lambda \to \mathbb{C}/\Lambda$ are inverses.)

For some of what follows Λ will be "tacit" (not explicitly mentioned). E_Λ was called the (Weierstraß form) elliptic curve associated to Λ. We will now write simply "E" for simplicity.

II. Group law on an elliptic curve

Let $p, q \in E$, and consider the two lines



each of which intersects E in 3 points "with multiplicity": parametrize L_i by $t \mapsto (x(t), y(t))$ and factor the cubic polynomial $y(t)^2 - 4x(t)^3 + g_2 x(t) + g_3$.

<u>Remark</u>// We consider a line to go through the "point at ∞" $\Leftrightarrow$ it is vertical. (The right way to think about this is in projective space, where the equation is $Y^2Z = 4X^3 - g_2XZ^2 - g_3Z^3$, $[0:1:0]$ is the point at ∞, and a "vertical" line is one given in the form $AX + CZ = 0$.) So the x & y coordinates for $p*q$ and $p+q$ are related by $(x,y) \leftrightarrow (x,-y)$. //

Note that for any p, $\infty + p = p$; and clearly also $p+q = q+p$. But so far, this isn't <u>yet</u> a group structure on E — just a binary operation.

$\boxed{\text{Theorem 2}}$ $\left.\begin{array}{l} \underline{\wp(u_1) + \wp(u_2) = \wp(u_1+u_2)} \\ \text{and} \\ \underline{u(p) + u(q) \underset{\Lambda}{\equiv} u(p+q)} \end{array}\right\}$ $\underline{\forall\, u_1, u_2, p, q}$.

<u>Hence "+" on E defines a group law, making $\wp$ and u into isomorphisms of abelian groups.</u>

Sketch: We only need to prove <u>one</u> of the formulas ($\Rightarrow$ equivalence [under $\mathbb{C}/\Lambda \cong E_\Lambda$] of binary operations $\Rightarrow$ other formula). Take $p, q \in E$.

Let f_1 & f_2 be the linear polynomials defining L_1 & L_2, and set $f := f_1/f_2$. This is a meromorphic function "on E" with order at a point $r \in E$ given by the order of intersection of L_2 & E minus the order of intersection of L_1 & E. The situation is

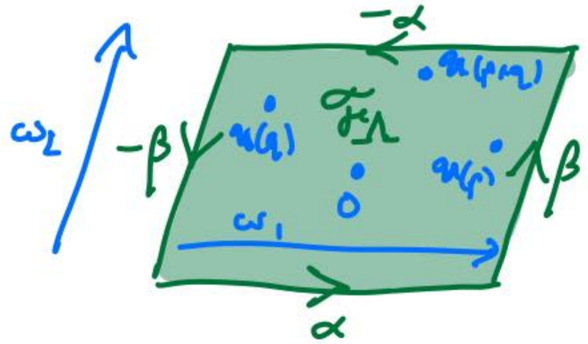
point of E	$p+q$	p	q	∞
order of f	1	-1	-1	1

and so its pullback $f \circ \wp$ to $\mathbb{C}/\Lambda$ has

point of $\mathbb{C}/\Lambda$	$u(p+q)$	$u(p)$	$u(q)$	0
order of $f\circ\wp$	1	-1	-1	1

and clearly $f \circ \wp$ is an elliptic function.

Consider a fundamental domain for $\mathbb{C}/\Lambda$



u = complex coordinate on this

and write

$$u(p+q) - u(p) - u(q) (+\,0) = \frac{1}{2\pi i}\int_{\partial\mathcal{F}_\Lambda} u \cdot d\log(f\circ \wp)$$

$$= \underbrace{\left\{\frac{-1}{2\pi i}\int_\alpha d\log(f\circ\wp)\right\}}_{\in\mathbb{Z}}\cdot\omega_2 + \underbrace{\left\{\frac{1}{2\pi i}\int_\beta d\log(f\circ\wp)\right\}}_{\in\mathbb{Z}}\cdot\omega_1$$

$$\in \Lambda.$$

since $f\circ\wp$ is elliptic (biperiodic)

$\square$

To make the theorem explicit, write

- $j: E \to E$ for $(x,y)\mapsto(x,-y)$
- $p = \wp(u_1) = (\wp(u_1), \wp'(u_1))$, $\quad q = \wp(u_2) = (\wp(u_2), \wp'(u_2))$
- $p * q = j(p+q) = j(\wp(u_1)+\wp(u_2))$

$$\underset{\text{Theorem}}{=} j(\wp(u_1+u_2)) = j(\wp(u_1+u_2), \wp'(u_1+u_2))$$

$$= (\wp(u_1+u_2), -\wp'(u_1+u_2)).$$

Now, $p, q,$ & $p*q$ are collinear, so their projective representations (i.e. replace (x,y) by $[1:x:y]$ and "∞" by $[0:1:0]$) are coplanar vectors in $\mathbb{C}^3$:

First Addition Theorem

$$\boxed{0 = \begin{vmatrix} 1 & 1 & 1 \\ \wp(u_1) & \wp(u_2) & \wp(u_1+u_2) \\ \wp'(u_1) & \wp'(u_2) & -\wp'(u_1+u_2) \end{vmatrix}}$$

This is the analogue for biperiodic functions of the standard trigonometric angle-addition formulas (or the basic relation $\exp(u_1+u_2) = \exp(u_1)\exp(u_2)$). Since $\wp'(u) = \pm\sqrt{4\wp(u)^3 - g_2\wp(u) - g_3}$, it really does express $\wp(u_1+u_2)$ in terms of $\wp(u_1)$ & $\wp(u_2)$.

For the next "addition theorem", we'll make this even more explicit. Write

$$L_1 = \{ax+b = y\} \subset \mathbb{C}^2,$$

and intersect with E by substituting:

$$0 = 4x^3 - g_2 x - g_3 - (\underbrace{ax+b}_{y})^2 = 4(x-x(p))(x-x(q))(x-\underbrace{x(p+q)}_{=x(p*q)})$$

$$\Longrightarrow \quad a^2 = 4\left(x(p) + x(q) + x(p+q)\right).$$

But since a is the slope of L_1, we have also:

$$a = \frac{y(q)-y(p)}{x(q)-x(p)}$$

$$\Longrightarrow \boxed{x(p+q) = \frac{1}{4}\left(\frac{y(q)-y(p)}{x(q)-x(p)}\right)^2 - x(p) - x(q)} \qquad (*)$$

So from $u(p)+u(q) \underset{\Lambda}{\equiv} u(p+q)$ we get (writing $Q(x) := 4x^3 - g_2 x - g_3$) :

Second Addition Theorem

$$\boxed{\int_\infty^{x_1} \frac{dx}{\sqrt{Q(x)}} + \int_\infty^{x_2} \frac{dx}{\sqrt{Q(x)}} \underset{\Lambda}{\equiv} \int_\infty^{\frac{1}{4}\left(\frac{\sqrt{Q(x_1)}-\sqrt{Q(x_2)}}{x_1-x_2}\right)^2 - x_2 - x_1} \frac{dx}{\sqrt{Q(x)}}}$$

which is a nontrivial functional equation for the elliptic integral $\int_\infty^{(\cdot)} \frac{dx}{\sqrt{Q(x)}}$ (in the same vein as $\log(x_1)+\log(x_2) \underset{2\pi i\mathbb{Z}}{\equiv} \log(x_1 x_2)$ or $\arcsin(x_1) + \arcsin(x_2) \underset{2\pi\mathbb{Z}}{\equiv} \arcsin\left(x_1\sqrt{1-x_2} + x_2\sqrt{1-x_1}\right)$). Also note that $(*)$ appears in this week's HW (with $\wp, \wp'$ standing in for x, y), and our approach to it here gives meaning in terms of the group law on E.