Lecture 26: More on modular forms

I. Dedekind’s eta function

Recall that we defined
\[ \eta(z) := q^{1/24} \prod_{n=1}^{\infty} (1-q^n) \in \text{Hol}(\mathbb{C}) \]

where \( q := e^{2\pi iz} \) has \( 0 < |q| < 1 \) for \( z \in \mathbb{C} \). Taking \( \log \),
\[ \frac{1}{12} \pi^2 z^2 - \log \eta(z) = \sum_{k=1}^{\infty} \frac{1}{k} q^{kz} \]
\[ = \sum_{k=1}^{\infty} \frac{1}{k} \frac{q^k}{1-q^{kz}}. \]

Set \( h_n(z) := \cot((n+\frac{1}{2})\pi z) \cot((n+\frac{1}{2})\pi z)/2 \). This has:

- Simple poles at \( z = \frac{1}{2k} + \frac{\pm k\pi i}{2} \) with residues \( \frac{1}{\pi k} \cot(\frac{\pm k\pi z}{2}) \) (\( k = 1, 2, \ldots \)) resp. \( \frac{1}{\pi k} \cot(\frac{\pi k z}{2}) \)
- 3rd order pole at \( z = 0 \) with residue \( -\frac{1}{3}(2z+2^{-1}) \)

Using the contour \( C \) and the relation
\[ -\frac{e^{iz} + e^{-iz}}{e^{iz} - e^{-iz}} = (1 + \frac{2}{e^{2iz} - 1}) \]
we therefore obtain:

\[ n = \text{poles of } h_n \]
\[
\pi i \frac{2 + 2^{-1}}{12} + \frac{1}{8} \int_C h_n(x) \, dx = \frac{1}{2} i \sum_{k=1}^{\infty} \left( \cot \left( \frac{\pi k x}{\tau} \right) + \cot \left( \frac{\pi k}{\tau} \right) \right)
\]

Using the fact that \( \chi_n(x) \) is uniformly bounded on \( C \) as \( n \to \infty \), with limits

we have

\[
\lim_{n \to \infty} \int_C h_n(x) \, dx = \left( \int_1^\tau - \int_{\tau}^1 + \int_0^{1/2} - \int_{1/2}^0 \right) \frac{dx}{x}
\]

\[
= 4 \log \tau - 2 \log(-1) = 4 \log(\tau i)
\]

Putting this together with (\#) and (\#\#), and setting

\[
\gamma = \frac{1}{2 \pi i \tau}
\]

\[
\pi i \frac{2 + 2^{-1}}{12} + \frac{1}{2} \log \frac{\tau}{i} = \sum_{k=1}^{\infty} \frac{1}{k} \frac{\gamma_k}{1 - \gamma_k} - \sum_{k=1}^{\infty} \frac{1}{k} \frac{\gamma_k}{1 - \gamma_k}
\]

\[
\Rightarrow \frac{1}{12} \pi i \tau - \log \eta(\tau) - \frac{1}{\tau} \pi i (-2^{-1}) + \log \eta(-\tau^{-1})
\]

\[
\Rightarrow 24 \log \eta(-\tau^{-1}) = 24 \log \eta(\tau) + \log \tau^4
\]

\[
\Rightarrow \eta \left( \frac{-1}{\tau} \right)^{24} = \tau^{12} \eta(\tau)^{24}. \text{ So } \eta^{24} \text{ is a cusp form of weight 12 w.r.t. } \Gamma \text{ (see Example 3 of Lecture 25).}
\]
II. Ring structure

At the end of Lecture 16, we showed that

\[(\#) \quad j(5_6) = 0, \quad j(i) = 1.\]

Another way to see this is by using the Proposition (Lect. 25, p. 7). For \(E_7\), we clearly must have a simple zero at \(5_6\); while for \(E_6\), we must have one at \(i\). (\#) follows.

Now here is a really amazing fact about \(E_7\) & \(E_6\).

**Theorem 1** \[\bigoplus_k M_k(\Gamma) = C[E_7, E_6].\] That is,

\(E_7\) and \(E_6\) jointly generate the "graded ring of modular forms".

**Proof:** Let \(P(X,Y) \in C[X,Y]\) be a polynomial with

\[P(f_1(z), f_2(z)) = 0\]

where \(f_1, f_2\) are modular forms of the same weight.

By considering the weights, we see that

\[(\#\#) \quad P_d(f_1, f_2) = 0\]

for each homogeneous component \(P_d\) of \(P\). But

\[\frac{P_d(f_1, f_2)}{f_2^d} = P(f_1/f_2)\]
for some $p(t) \in \mathbb{C}[t]$. Since $p$ has only finitely many roots, we can only have $(\#\#)$ if $f_1/f_2$ is a constant.

Consider the special case $f_1 = E_4^3$, $f_2 = E_6^2$. If we had $E_6^2 = \lambda E_4^3$ for some $\lambda \in \mathbb{C}^*$, then $f = E_6/E_4$ would satisfy $\lim_{t \to 0} f(t) = 1$ and $f^2 = \frac{E_6^2}{E_4^3} = \lambda E_4$, hence $0 \neq f \in M_2(\Gamma)$. But [Cor.1, lect. 25] says that $\dim M_2(\Gamma) = 0$. So $E_4^3$ and $E_6^2$ can have no polynomial relation, and are therefore algebraically independent.

Furthermore, any algebraic relation $Q(E_4, E_6) = 0$ breaks up into “homogeneous” components — summands with all terms of the same weight: $Q_k(E_4, E_6) = 0$. Any two terms in such a summand, say $c_i E_4^a E_6^b$ and $c_j E_4^{a'} E_6^{b'}$, have $4a + 6b = k = 4a' + 6b'$. But then

$$2a + 3b = 2a' + 3b' \Rightarrow a \equiv a' \mod (3), \quad b \equiv b' \mod (2).$$

$$\Rightarrow \text{algebraic relation on } E_4^2, E_6^2.$$

Finally, [Cor.1, lect. 25] says

$$\dim M_k(\Gamma) \leq \begin{cases} \left\lfloor \frac{k}{12} \right\rfloor + 1, & k \not\equiv 2 \mod 12 \\ \left\lfloor \frac{k}{12} \right\rfloor, & k \equiv 2 \mod 12. \end{cases}$$
For $k = 12n$, the weight $k$ part of $C[E_4, E_6]$ is

$$C \langle E_4, E_4^3 E_4, E_4^3 E_6, E_4^4 E_6, \ldots, E_6^{2n} \rangle$$

which has dimension $n+1$. For $k \in \{0, 1, 9, 18, 10\}$ it is $C$, $C \langle E_4 \rangle$, $C \langle E_6 \rangle$, $C \langle E_4^3 \rangle$, and $C \langle E_4 E_6 \rangle$. Putting these two calculations together gives $\dim(C[E_4, E_6]_k) \geq \text{RHS}(!)$

So in

$$\text{RHS}(!) \geq \dim M_k(\Gamma) \geq \dim C[E_4, E_6]_k \geq \text{RHS}(!)$$

all inequalities are equalities and we are done.

Example 1: Comparing constant terms in Fourier expansions, we have (as a consequence of the Theorem)

$$E_4^2 = E_4, \quad E_4 E_6 = E_{10},$$

$$E_6 E_4 = E_4 E_{10} = E_{14}.$$  

This leads to some surprising number-theoretic identities. For instance,

$$E_4^2 = 1 + 480 \sum_{n=1}^{\infty} \sigma_3(n) q^n + 480 \cdot 120 \sum_{n=1}^{\infty} \left( \sum_{m=1}^{n} \sigma_3(m) \sigma_3(n-m) \right) q^n$$

and

$$E_9 = 1 + 480 \sum_{n=1}^{\infty} \sigma_3(n) q^n$$

so

$$\sum_{m=1}^{n-1} \sigma_3(m) \sigma_3(n-m) = \frac{\sigma_3(n) - \sigma_3(n)}{120}$$.