Lecture 3: Extension to the boundary

The potential ugliness of the boundary of a simply-connected region \( \Omega \subseteq \mathbb{C} \) was historically a major obstacle to the late 19th century attempts (via "Dirichlet Principle"/potential theory) to prove the full RMT.

What we’ll discuss today are situations in which the map \( f: \Omega \rightarrow D_1 \) produced by RMT admits a continuous resp. analytic extension to the closure \( \overline{\Omega} \) (or part of it).

At the end I’ll give one more proof of the Riemann Mapping Theorem itself, one with a more constructive & dynamical flavor. For reference, recall the statement:

\[
\text{RMT: For } \Omega \subseteq \mathbb{C} \text{ a simply connected region, there exists a biholomorphic mapping } f: \Omega \rightarrow D_1.
\]
I. A topological result

Let $R, R' \subseteq \mathbb{C}$ be regions.

**Definition** A sequence $\{z_n\} \subseteq R$ (resp. path $Y : [0,1] \to R$)

is said to **approach the boundary of $R$ iff** for each compact $K \subseteq R$ there exists $N \in \mathbb{N}$ (resp. $\epsilon > 0$) such that $\{z_n\}_{n \geq N} \subseteq R \setminus K$ (resp. $Y((1-\epsilon, 1)) \subseteq R \setminus K$).

**Proposition** Let $F : R \to R'$ be a homeomorphism, and $\{z_n\}$ (or $Y(t)$) $\to \partial R$. Then $\{F(z_n)\}$ (or $F(Y(t))$) $\to \partial R'$.

**Proof:** Let $K' \subseteq R'$ be compact. Since $F$ is a homeomorphism, $F^{-1}(K')$ is compact.

But then some tail of $\{z_n\}$ or $Y(t)$ stays outside $F^{-1}(K')$, and so the corresponding tail of $\{F(z_n)\}$ or $F(Y(t))$ avoids $K'$.
II. Extension via Schwarz reflection

Let $\mathcal{D}$ be as in the RMT.

**Definition.** An analytic arc is a map
\[ \gamma : (a, b) \rightarrow \mathbb{C} \]
which is 1-to-1 and real-analytic, with $\gamma'$ nowhere 0.
[As usual we shall write $\gamma$ for the image as well.]

(5) A (free, one-sided) analytic boundary arc of $\mathcal{D}$ is an analytic arc $\gamma \subset \partial \mathcal{D}$ with (complex-) analytic extension
\[ \tilde{\gamma} : \Delta \rightarrow \mathbb{C} \]
where:
- $\Delta \subset \mathbb{C}$ is open simply-connected, with $\Delta \cap \mathbb{R} = (a, b)$
- $\Delta$ is symmetric under complex conjugation
- $\tilde{\gamma}^{-1}(\mathbb{R}) = \Delta \cap \mathbb{R}$.

Note that the extension $\tilde{\gamma}$ is unique, and defined by the same power-series defining $\gamma$.

Now let $f : \mathcal{D} \rightarrow D_1$ be as in the RMT.

**Theorem.** If $\gamma \subset \partial \mathcal{D}$ is an analytic boundary arc, there is an extension $f \in Hol(\mathbb{R} \cup \gamma)$ with $f \mid \gamma$ an analytic boundary arc of $\partial D_1$. 
Proof: We assume \( \tilde{y}' \) nowhere 0

- \( \tilde{y} \) 1-to-1

by shrinking \( \Delta \), so that \( \tilde{y} : \Delta \rightarrow \tilde{y}(\Delta) \) is a conformal isomorphism. It will suffice to extend

\[
F := f \circ (\tilde{y} |_{\Delta \cap \mathbb{R}}) \in \text{Hol}(\Delta \cap \mathbb{R})
\]

to \( \tilde{F} \in \text{Hol}(\Delta) \), since

\[
\tilde{f} := \tilde{F} \circ \tilde{y}^{-1} \in \text{Hol}(\tilde{y}(\Delta))
\]

then agrees with \( f \) on \( \tilde{y}(\Delta \cap \mathbb{R}) \).

Shrinking \( \Delta \) further if necessary, we may assume 0 \( \notin F(\Delta \cap \mathbb{R}) \), so that \( i \log F \) is defined on \( \Delta \cap \mathbb{R} \). Since for any \( \{z_j\} \subset \Delta \cap \mathbb{R} \) approaching \( \mathbb{R} \) we have \( F(z_j) \rightarrow 2\pi i \) by the Proposition above (9.I),

\[
\text{Im} \ (i \log F(z_j)) = \log |F(z_j)| \rightarrow 0.
\]

By Schwarz reflection, we may extend \( i \log F(z) \)

- by its limit on \((a, b) = \Delta \cap \mathbb{R})
- by \( i \log F(\overline{z}) \) on \( \Delta \cap (-\infty, a) \)

to a holomorphic function on \( \Delta \). Taking \( \exp(-i(\cdot)) \)

yields the extension of \( F \) itself (which satisfies \( i \log F(z) = -i \log F(\overline{z}) = i \log \left( \frac{1}{F(\overline{z})} \right) \Rightarrow F(z) = \overline{F(\overline{z})} \)).
It remains to check that \( f \circ y = \tilde{F}\big|_{y(a)} \) is 1-to-1.

If \( \tilde{F}'(x_0) = 0 \) for any \( x_0 \in (a,b) \), then \( \tilde{F} \) would have to map not just \( x_0 \) but curves with tangent \( x_0 + ie^{in\pi} \) (for some \( n \geq 2 \)) to \( \partial D_2 \), impossible since \( \tilde{F}(\Delta n h) \cap \partial D_1 = \emptyset \) (and such curves would intersect \( \Delta n h \)). But if \( \tilde{F}'(x_0) \neq 0 \), then

\[
0 > \frac{\partial \log|\tilde{F}|}{\partial y} \bigg|_{x_0} = -\frac{\partial \log \tilde{F}}{\partial x} \bigg|_{x_0} \quad \text{ca. eqs.}
\]

\( \tilde{F}\big|_{y(a)} \) maps strictly counterclockwise in \( \partial D_1 \) as \( x \) increases, hence is 1-to-1.

Note: If you forgot about Schwarz reflection, it's on pp. 172-3 of Ahlfors.
III. Caratheodory's Theorem

The next result concerns the case where the boundary of $\Omega$ is a continuous Jordan curve; i.e.
there is a $C^0$ map

$$\gamma: \mathbb{S}^1 \to \partial \Omega$$

that is 1-1 onto, hence a homeomorphism. ($\mathbb{S}^1$ is taken to be the bounded component of $\mathbb{C} \setminus \gamma(\mathbb{S}^1)$, and is a bounded, simply connected region.)

**Theorem**  Let $\varphi: D_1 \to \Omega$ be a conformal isomorphism, with $\Omega$ as above (bounded, simply-connected region, with $\partial \Omega$ $C^0$ Jordan). Then there exists $\Phi: \overline{D}_1 \to \overline{\Omega}$ $C^0$ and 1-to-1, such that $\varphi|_{D_1} = \Phi|_{D_1}$ — that is, $\varphi$ admits an extension to a homeomorphism of the (compact) closures.

The proof is long and is deferred to Lecture 4.

An obvious corollary of this Theorem (together with RMT) is that for $\Omega_1$, $\Omega_2$, Jordan-curve-boundary regions, $\exists$ homeomorphism $\overline{\Omega}_1 \cong \overline{\Omega}_2$ restricting to a conformal isomorphism $\Omega_1 \cong \Omega_2$. 
IV. The second proof of RMT

We can construct approximate mappings of a bounded simply-connected region \( R \) into \( D_1 \) in the sense of the following

**Lemma (Caratheodory):** \( \exists \{ f_n \} \subset \text{Hol}(R, D_1) \) s.t.  

1. \( f_n(P) = 0 \) (for some fixed \( P \in \mathcal{R} \))  
2. \( f_n(z) \) maps \( \mathcal{R} \) to a region \( \mathcal{R}_n \) in 1-1 fashion, with \( D_{r_n} \subset \mathcal{R}_n \subset D_1 \) (\( r_n \in (0,1) \)).  
3. \( r_n \to 1 \) as \( n \to \infty \).

**Somewhat heuristic** Proof: Take \( \mathcal{R}_0 = \mathcal{R} \), and define \( \mathcal{R}_1 := f_1(\mathcal{R}_0) \), where \( f_1(z) := \kappa \cdot (z - P) \) translates & dilates \( \mathcal{R}_0 \) to fit it inside \( D_1 \). Let \( r_1 := \text{radius of the largest } D_r \subset \mathcal{R}_1 \).

Some \( z_1 \in \partial D_{r_1} \) is not in \( \mathcal{R}_1 \) (since \( \mathcal{R}_1^c \) is closed and \( d(\mathcal{R}_1^c, \partial D_{r_1}) = 0 \)).

Now inductively define, given \( \{ z_n \in \partial \mathcal{R}_n \} \),
\[
\phi_{n+1} = \left( \phi_{n+1} \circ S^{-1} \circ \phi_n \right) \circ \phi_n.
\]

geometrically, \( \phi_{n+1} \circ S^{-1} \circ \phi_n \) pushes the point \( \mathbf{z}_n \) on the boundary of \( D_n \) to \( 0 \), takes square root, and pushes \( 0 \) out again, to a point at distance \( r_n^{1/2} \) from \( 0 \) (i.e. farther out). The circle of radius \( r_n \) is mapped to a lemniscate (Witch of Agnesi) of smallest radius \( = R(r_n) \), and clearly this is a lower bound for \( r_{n+1} \).

To compute \( r_{n+1} \), it is enough to consider
\[
\phi_{n+1}^{-1} \circ S^{-1} \circ \phi_n,
\]
which sends
\[
\mathbf{z}_n \rightarrow 0 \rightarrow \mathbf{z}_n \rightarrow -R(r_n) \rightarrow \sqrt{r_n},
\]
(\textit{It's up to you to check that the closest point indeed has phase } \pi \text{.)}
We have
\[
\sqrt{\frac{v_n(1-e^{i\theta})}{1-v_n e^{i\theta}}} = \frac{\sqrt{v_n} - R(v_n) e^{i\theta}}{1 - \sqrt{v_n} R(v_n) e^{i\theta}}
\]

\Rightarrow \sqrt{\frac{2v_n}{1+v_n^2}} = \frac{\sqrt{v_n} + R(v_n)}{1+\sqrt{v_n} R(v_n)}

\Rightarrow R(v_n) = \frac{\sqrt{v_n} (v_n-1) + \sqrt{2v_n(1+v_n^2)}}{1+v_n} \quad (\leq r_m).

(The point is that if the \( r_n \)-disk is contained in \( A_n \), then the lemniscate I drew has to be in \( R_{nm} \).)

So... locally at 0, \( R(r) \) has dominant term \((\sqrt{2}-1)\sqrt{r} (> r)\); thus for small \( r \) the function \( R(r) \) is increasing fast. Further, solving
\[
r = \frac{\sqrt{v_n} (v_n-1) + \sqrt{2v_n(1+v_n^2)}}{1+v_n}
\]
leads to \( r (v_n^2-1) (v_n-1)^{1/2} < 0 \)

meaning the graph of \( R \) looks like

\begin{center}
\includegraphics[width=0.5\textwidth]{graph.png}
\end{center}

so clearly \( r_n < R(v_n) \leq r_m \leq 1 \)

and \( r_n \to 1 \).
SECOND PROOF of RMT: We'll do this for bounded simply-connected \( S \) — obviously enough (cf. the proof of Lemma 2 in Lecture 2):

- The \( \{f_n\} \) produced by the lemma are uniformly bounded by 1; and so by Montel, some subsequence converges uniformly on (all) compact subsets. Since the limit function \( f \) is (in any such set) a uniform limit of analytic functions, it must be analytic. Furthermore, \( f(0) = 0 \).

- The same argument as in the 1st proof shows \( f \) is 1-1.

- So (as before) we must check \( f \) is onto \( D_1 \). This goes a little differently.

Take any \( w_0 \in D_1 \); \( w_0 \) lies in \( D_{1-\epsilon} \) for some \( \epsilon > 0 \). We may assume \( S \) is bounded and obtain that the limit \( F \) of \( \{f_n^{-1} \mid n \geq N\} \) (choose \( N \) s.t. \( r_N \geq 1-\epsilon \)) on \( D_{1-\epsilon} \) is analytic and 1-1 (by taking a subsequence and applying Montel and Hurwitz as above). Consider the compact subset \( F(D(w_0, \epsilon)) = \overline{V} \); since \( F \) is 1-1, the interior \( F(D(w_0, \epsilon)) \) is a neighborhood of \( F(w_0) \) (with compact closure \( \subset S \)), and
So contains all \( \{ f^{-n}(w_0) \} \) for \( n \geq M \geq N \). Also \( f \) is \( C^0 \) on some large image of \( I \).

The \( \{ f_n \} \) converge uniformly on the compact closure (and are continuous there), so that we may write

\[
W_0 = \lim_{n \to \infty} \left( f_n \circ f_n^{-1} \right)(w_0) = \lim_{n \to \infty} \lim_{m \to \infty} \left( f_n \circ f_m^{-1} \right)(w_0)
\]

\[
= \lim_{n \to \infty} f_n \left( \lim_{m \to \infty} f_m^{-1}(w_0) \right) = \lim_{n \to \infty} f_n \left( f(w_0) \right) = f(F(w_0)) = f(W_0)
\]

So that indeed \( f \) hits \( W_0 \).