Lecture 32: Schlicht functions II

Recall (for reference) from Lecture 31 the

A2 Theorem ( Bieberbach, 1916)
\[ f \in S \implies |a_2| \leq 2, \]
with equality iff \( f \) is "a rotation of the Köbe function", i.e.
\[ f(z) = e^{-i\alpha} K(e^{i\alpha} z) \]
for some \( \alpha \in \mathbb{R} \).

I. The \( \frac{1}{q} \) theorem

Theorem 1 (Köbe, 1907)

1. \( f \in S \implies f(0) \neq D_{iy} \)
2. If (for \( f \in S \)) \( f(0) = D_{iy} \), then
\[ f(z) = e^{-i\alpha} K(e^{i\alpha} z) \]
for some \( \alpha \in \mathbb{R} \).

Proof: Let \( b \in \mathbb{C} \setminus f(0) \). Since \( b \neq 0 \),
\[ g(z) := \frac{f(z) - b}{f(0) - b} \text{ is holom. on } D, \quad \text{with } g(0) = 0 \text{ and } g'(0) = 1. \]
Since \( g = \text{FLT of } f \), \( g \) is \( 1 \); so \( g \in S \).
Write \( g = \frac{\frac{2}{5} + a_1 \frac{2}{5} + O(\frac{2}{5})}{1 - \frac{1}{5} \frac{2}{5} + O(\frac{2}{5})} \)
\( = \left( 1 + a_1 \frac{2}{5} + O(\frac{2}{5}) \right) (1 + \frac{2}{5} + O(\frac{2}{5})) \)
\( = 1 + a_1 \frac{2}{5} + O(\frac{2}{5}) \).

"A_2 theorem" \( \Rightarrow \left| a_1 + \frac{1}{5} \right| \leq 2 \text{ if } \left| a_1 \right| \leq 2 \)
\( \Rightarrow \frac{1}{|b|} = \left| (\frac{5}{6} + a_2) - a_2 \right| \leq \frac{5}{6} + |a_2| + |a_1| \leq 4 \)
\( \Rightarrow f(D) > D_{14} \text{ (as } b \text{ was an arbitrary point not in } f(D) \text{).} \)

Now, for the extremal case: if \( f \) is not a rotation of \( K \),
then the \( A_2 \) theorem \( \Rightarrow \)
\( \Rightarrow |a_1| = 2 - \epsilon \text{ for some } \epsilon \in (0, 2] \)
\( \Rightarrow \frac{1}{|b|} \leq 2 + (2 - \epsilon) = 4 - \epsilon \)
\( \Rightarrow |b| \geq \frac{1}{4 - \epsilon} > \frac{1}{4} \Rightarrow f(D) > D_{14} \).

There is a nice application to Smale's conjecture (end of lect. 30): \( \text{let } P(x) = x + \sum \frac{a_k}{k!} x^k \) (note \( P(0) = 0, P'(0) = 1 \),
\( \{z_1, z_2, \ldots, z_n\} = 0 \) s. \( \mathbb{R} \), \( \{w_1, \ldots, w_n\} = \) their images under \( P \),
\( \delta(P) := \min_{1 \leq j \leq n} \frac{|w_j|}{|z_j|} \). Conjecture is that \( \delta(P) \leq \frac{n}{n+1} \) (known for \( n \leq 3 \)).

**Theorem 2 (Cerwinski, Kowalski-Smale)** \( \delta(P) \leq 4 \).

**Proof:** [Note: read appendix on covering maps first.]
First, \( \mathcal{S}_2 = \mathbb{C} \setminus \{w_1, \ldots, w_n\} \)

\[
P' \uparrow \\
\mathcal{S}_1 = P^{-1}(\mathcal{S}_2) \subset \mathbb{C} \setminus \{z_1, \ldots, z_m\}
\]

is an analytic covering map, since zeros of \( P' \) are omitted. Wolog assume \( M := |w_1| < |w_2| < \cdots < |w_n| \).

Applying the lifting theorem to

\[
\begin{array}{c}
\mathcal{S}_1 \\
\downarrow P \\
\mathcal{S}_2
\end{array}
\]


\[
\begin{array}{c}
\mathbb{D}_M \\
\hookrightarrow
\end{array}
\]

yields \( f \) s.t. \( P \circ f = i \). Note that \( i \) injective \( \Rightarrow \) \( P \) injective.

Moreover, we can choose \( f \) so that 0 is sent to any point in \( P^{-1}(0) \); so choose \( f(0) = 0 \).

Set \( f_1(e) := \frac{f(e M)}{M} \in \text{Sh}(D) \). We have

\[
1 = i'(0) = P'(f(0)) \cdot f'(0) = P'(0) \cdot f'(0) = f'(0)
\]

\[\Rightarrow f'_1(0) = f'(0) = 1 \quad f_1(0) = 0\]

\[\Rightarrow f_1 \in \mathcal{S}.
\]

Since \( \frac{z_j}{M} \in \mathcal{S}_1 \), \( f(D_M) \) contains no \( \frac{z_j}{M} \)

\[\Rightarrow f_1(D) := \left( \frac{f(D_M)}{M} \right) \text{ contains no } \frac{z_j}{M}.
\]

By \( \text{Köbe} \), \( f_1(D) \supset D_{M_1} \), and so each \( \frac{z_j}{M_1} \geq \frac{1}{M} \).

In particular, \( |z| \geq \frac{M_1}{q} = \frac{|w|}{q} \Rightarrow \text{min } \frac{|z|}{|z_j|} \leq \frac{|w|}{|z_j|} \leq q \).
II. Distortion

As usual $D = D_1$.

Lemma 1: Given

\[ Y : [0, 1] \rightarrow D \text{ path from } 0 \text{ to } 2. \]

\[ g \in C^0(0, 1), \ g \geq 0 \]

Then $\int_Y g(|x_1|) \, |dx_1| \geq \int_0^1 g(r) \, dr$.

Proof: Although this isn’t $C^0$, it’s enough to prove this for $g = \chi_{(a, b)}$ (characteristic function), with $(a, b) \subset (0, 1)$. We have

\[ \int_Y g(|x_1|) \, |dx_1| = \int_{y_{a,b}} |d \omega| = \int_E |Y'(t)| \, dt \quad (\dagger) \]

Let $t_2 := \min \{ t \in (0, 1) \mid |Y(t)| = b \}$

\[ t_1 := \max \{ t \in [0, t_2) \mid |Y(t)| = a \} \]

Then $[t_1, t_2] \subset E \Rightarrow (\dagger) \geq \int_{t_1}^{t_2} |Y'(t)| \, dt$

\[ \geq \left| \int_{t_1}^{t_2} Y'(t) \, dt \right| \]

\[ = |Y(t_2) - Y(t_1)| \]

\[ \geq |Y(c_a) - Y(c_b)| \]

\[ = |b - a| \]

\[ = \int_0^1 g(r) \, dr \]
Recall the Köbe function
\[ K(z) := \frac{z}{(1-z)^2} \]
with
\[ K'(z) = \frac{(1+z)}{(1-z)^3}. \]
We have
\[
\sup_{|\theta| = \pi} |K(e^{i\theta})| = \frac{r}{(1-r)^2} = K(r), \quad \inf_{|\theta| = \pi} |K(e^{i\theta})| = \frac{r}{(1+r)^2} = K(-r),
\]
\[
\sup_{|\theta| = \pi} |K'(e^{i\theta})| = \frac{r}{(1-r)^3} = K'(r), \quad \inf_{|\theta| = \pi} |K'(e^{i\theta})| = \frac{1-r}{(1+r)^3} = K'(-r).
\]

**Theorem 3 (Köbe, 1907; Bieberbach, 1916):** Given \( f \in \mathcal{S}, \quad r \in (0,1) \):

(a) We have \( \begin{align*}
(i) \quad & \frac{1-r}{(1+r)^2} \leq |f'(e^{i\theta})| \leq \frac{1+r}{(1-r)^2} \\
(ii) \quad & \frac{r}{(1+r)^2} \leq |f(e^{i\theta})| \leq \frac{r}{(1-r)^2}
\end{align*} \) \( \forall \theta \in \mathbb{D} \).

(b) If for some \( \theta \in \partial \mathbb{D}_r \), one of the four inequalities in (a) is an equality, then \( f \) is a rotation of \( K \).

**Proof:**

(a) **RHS of (i):** Write \( \tau_d(\theta) = \frac{\theta + d}{1 + d\theta} \), \( \theta \in \mathbb{D} \).

Set \( g(\theta) := \frac{f(\tau_d(\theta)) - f(\theta)}{(f \circ \tau_d)'(0)} \in \mathbb{D} \) \( (g \) is 1-1 because of the form \( \text{FLT} \circ \text{f FLT} \).

\[ f'(\theta) \tau_d'(\theta) \]
By the "Go To Theorem", \[ \log(\varepsilon) = 12\varepsilon - t. \]

Now \[ t_x(0) = 1 - H^2, \]

Thus \[ t_x(1) = \frac{1}{x+1}, \]

\[ x_0 = \frac{2}{2+1} = \frac{2}{3}, \]

Thus \[ t_x(1) = \frac{1}{2}, \]

But then, since \[ t_x(0) \leq \frac{1}{1-x^2}, \]

Changing \( x \) to \( 1-x^2 \) and multiplying by \( \frac{1}{1-x^2} \), we get

\[ t_x(0) \leq \frac{1}{4x^2} + \frac{1}{1-x^2} - 2x \]

\[ t_x(0) = 16 \varepsilon, \]

Thus \[ t_x(1) = \frac{1}{12} \varepsilon, \]

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for $|z| = r$ we have

\[
\left(-r \cdot \frac{K''(-r)}{K'(-r)} \right) \leq \text{Re} \left( \frac{z f''(z)}{f'(z)} \right) \leq r \cdot \frac{K''(r)}{K'(r)}
\]

\[\downarrow \text{take } t = r\]

\[
\frac{d}{dr} \text{Re} \left( \log f'(r) \right) = \text{Re} \left( \frac{f''(r)}{f'(r)} \right) \leq \frac{K''(r)}{K'(r)} = \frac{d}{dr} \log K'(r).
\]

Since the logs are 0 at $r = 0$ (why?), integrating

\[
\int_0^r \Rightarrow \text{Re} \left( \log f'(r) \right) \leq \log K'(r).
\]

\[\Rightarrow |f'(r)| \leq K'(r). \quad (\dagger)
\]

Integrating one more (using that $f(0) = K(0) = 0$) yields

\[
|f(r)| \leq \int_0^r |f'(t)| \, dt \leq K(r). \quad (\ddagger\ddagger)
\]

Given $\Theta_0$, let $f_i(z) = e^{-i\Theta_0} f(z e^{i\Theta})$ ; then

\[
|f(re^{i\Theta})| = |f_i(r)| \leq K(r) \quad \Rightarrow \text{done } u/RHS \text{ of (iii) (iv)}.
\]

\[
|f'(re^{i\Theta})| = |f_i'(r)| \leq K'(r)
\]

\[\log (\dagger)(\ddagger\ddagger)
\]

\[\text{(a) [LHS + (ii)](iii) : } \text{The left-hand part of (iii) gives}
\]

\[
\frac{d}{dr} \log K'(-r) = -\frac{K''(-r)}{K'(-r)} \leq \text{Re} \left( \frac{f''(r)}{f'(r)} \right) = \frac{d}{dr} \text{Re} \log f'(r)
\]

\[\downarrow \text{take } t = r\]

\[
\log K'(-r) \leq \text{Re} \log f'(r) = \log |f'(r)|
\]

\[\Rightarrow K'(-r) \leq |f'(r)|, \text{ same for } f_i,
\]
hence \( K'(-r) \leq |f'(re^{i\theta})| \Rightarrow \text{LHS of (a)(i)}. \)

Next, take \( z_0 = r_0e^{i\theta} \in \mathbb{D} \); then
\[
|K(-r_0)| \leq |K(-1)| = \frac{1}{4}.
\]

**CASE 1:** \( |f(z_0)| \geq \frac{1}{4} \) ; then
\[
|f(z_0)| \geq |K(-r_0)| (\Rightarrow \text{LHS of (a)(ii)}).
\]

**CASE 2:** \( |f(z_0)| < \frac{1}{4} \) ; then the radial segment \([0, f(z_0)]\)
belongs to \( \mathbb{D}_{u_2} \), hence (by Köbe \( \frac{1}{4} \) theorem) to \( f(\mathbb{D}) \).

Set \( Y := f^{-1}(\mathbb{D}_{u_2}, f(z_0)) \) (set part from 0 to \( z_0 \) \( \subset \mathbb{D} \)).

Then
\[
|f(z_0)| = \int_{0}^{f(z_0)} |dw| = \int_{Y} |f'(z)| \, dt
\]
\[
\geq \int_{Y} K'(-1, 1) \, dt
\]

**Lemma 1:**\( \Rightarrow \int_{0}^{r_0} K'(-1, 1) \, dt \)
\[
= -K(-r_0) \Rightarrow \text{LHS (a)(ii)}.
\]

(b): Apply (K) with \( z = r \):
\[
\frac{2r^2 - 4}{1 - r^2} \leq \Re \frac{f''(r)}{f'(r)} \leq \frac{2r^2 + 4}{1 - r^2}
\]
(\#)

i.e.
\[
\frac{1}{2r} \log K'(r) \leq \frac{1}{2r} \log |f''(r)| \leq \frac{1}{2r} \log K'(r).
\]

If \( |f'(r_0)| = K'(r_0) \) for some \( r_0 \in (0, 1) \), then clearly

RHS (\#) is an equality for all \( r \in (0, r_0) \). Taking \( r \to 0 \),
we get \( 2 \Re(a_2) = 1 \) \( \Rightarrow \) \( |a_2| = 2 \Rightarrow f \) = rotation of \( K \).
In fact, $\Re(a_2) = |a_2| = 2 \Rightarrow a_2 = 2 \Rightarrow f = K$.

If $|f(r_0)| = K(r_0)$ then $|f'(r)| = K'(r) \forall r \in (0, r_0)$, which again (by same argument) $\Rightarrow f = K$.

If $|f'(r_0)| = K(-r_0)$ or if $|f(r_0)| = -K(-r_0)$, an analogous argument $\Rightarrow f(x) = -K(-x)$.

**Appendix: Covering Spaces**

Let $\Omega_1 \xrightarrow{F} \Omega_2$ be a continuous mapping of topological spaces. $F$ is called a covering map if it is surjective and a local homeomorphism. (More precisely, one should require that there is a "discrete space" $\mathcal{D}$ and for each $p \in \Omega_2$ a nbhd. $(p)U \subset \Omega_2$ s.t. $F^{-1}(U) = \bigsqcup_{g \in \mathcal{D}} U_g$ with $F|_{U_g} : U_g \rightarrow U$ local homeo.) When $F$ is an analytic map of regions (or Riemann surfaces), this will mean that a sufficiently small disk $D$ about each point in $\Omega_2$ has preimage equal to a (nonempty) disjoint union of disk-like blobs in $\Omega_1$, each of which $F$ maps 1-1 conformally onto $D$.

Equivalent conditions are that the derivative $F'$ be everywhere nonzero ("$F$ étale"), or that there be no ramification points (when $z \mapsto z^k$ locally); this is enough because of the inverse mapping theorem.
Lifting Theorem: Given \( \Omega_0 = \text{simply connected region} \) and \( g : \Omega_0 \rightarrow \Omega_2 \) holomorphic, there exists a (holomorphic) "lifting" map \( G : \Omega_0 \rightarrow \Omega_1 \) such that \( F \circ G = g \).
Moreover, given \( t_0 \in \Omega_0 \) and any \( w_0 \in F^{-1}(g(t_0)) \), we may arrange that \( G(t_0) = w_0 \).

Sketch: Taking a sufficiently small ball \( B \) about \( t_0 \in \Omega_0 \), and composing \( g \) with a local branch of \( F^{-1} \) on \( g(B) \), give a germ \( G_0 \) at \( t_0 \). Any path on \( \Omega_2 \) lifts (uniquely, after fixing the initial point's lift) to a path on \( \Omega_1 \), which may not be closed even if the one on \( \Omega_2 \) is.

In particular, taking a path in \( \Omega_0 \) from \( t_0 \), we can cover it with balls sufficiently small that their \( g \)-images have homeomorphic preimages under \( F \) covering the lifted path. In this way one gets an analytic continuation of \( G_0 \) along all paths in \( \Omega_0 \), and since \( \Omega_0 \) is simply connected, one is done by the monodromy theorem.