Lecture 33: Green's functions on Riemann surface I

I. Riemann surfaces and universal covers

**Definition**

(i) Let \( M \) be a connected Hausdorff space, 
\[ \mathcal{U} = \{ U_x \mid U_x \subset M \} \] 
an open cover of \( M \),
\[ \tilde{z}_x : U_x \to \mathbb{C} \] 
continuous 1-1 maps such that the
homeomorphisms \( \tilde{z}_\beta \circ \tilde{z}_x^{-1} : \tilde{z}_x(U_x \cap U_\beta) \to \tilde{z}_\beta(U_x \cap U_\beta) \) are
(hi) holomorphic. Then \( \mathcal{A} = \{(\tilde{z}_x, U_x)\} \) is a conformal atlas,
\((M, \mathcal{A})\) a Riemann surface (usually denoted "\( M \)").

(ii) A function \( f \) on \( M \) is holomorphic
\[ \iff \{ f \circ \tilde{z}_x^{-1} \} \text{ are holomorphic.} \]

(iii) A map \( F : M \to M' \) of RS is holomorphic
\[ \iff \{ \tilde{z}_\beta \circ F \circ \tilde{z}_x^{-1} \} \text{ are holomorphic.} \]

This turns out to be equivalent to the "multi-valued analytic function element" approach, simply because one can always produce a conformal pair of holomorphic maps \( M \to \mathbb{P}' \) such that \( M \to \mathbb{P}' \times \mathbb{P}' \) is an immersion (with finitely many transverse crossings in the image).
Every RS $M$ has a universal cover $\tilde{M} \rightarrow M$, which is a simply connected covering space. One first constructs this topologically, then layers on the holomorphic structure. One may view $M \in \pi_1(\tilde{M})$ as the quotient of $\tilde{M}$ by a group of "deck transformations" (parametrizing branches of $M$ over $M$), acting properly discontinuously (so that the quotient is Hausdorff). This makes sense because the preimage of a sufficiently small neighborhood of $M$ is a tower of neighborhoods on $\tilde{M}$, and $\Gamma$ acts by holomorphic automorphisms of $\tilde{M}$.

Suppose now $\tilde{M} = \mathbb{C}$. Recall that $\text{Aut}(\mathbb{C})$ consists of transformations of the form $z \mapsto az + b$. This has 3 types of "properly discontinuous subgroup":

\[ g \in \Gamma \]

that is, every $x \in \tilde{M}$ has a nbhd. $U$ s.t. $gU \cap U \neq \emptyset \Rightarrow g = id$. To ensure the quotient is Hausdorff, we actually need a little more: for $x, x' \in \tilde{M}$ not in the same $\Gamma$-orbit, $\exists$ nbhds. $U, U'$ s.t. $gU \cap U' = \emptyset$ ($\forall g \in \Gamma$).

\[ g \in \Gamma \]

recall this means (roughly) that $\varphi$ is surjective (onto) and a local homeomorphism — no ramification allowed!!
• $\text{then supp. } \{0\} \Rightarrow M = \mathbb{C}$
• $\mathbb{Z} \langle \lambda \rangle \ (\lambda \in \mathbb{C}^*) \Rightarrow M = \mathbb{C} \left\{ 2 \exp \frac{\lambda \cdot \tau}{\exp} \right\}$
• $\mathbb{Z} \langle \lambda, \tau \rangle \ (\lambda, \tau \in \mathbb{R}) \Rightarrow M = \mathbb{C} \left\{ \lambda \cdot \tau \right\}$, all curve $E_{\lambda}$

Next suppose $\tilde{M} = \hat{\mathbb{C}}$ (i.e. $\mathbb{C}'$). We had $\text{Aut} \hat{\mathbb{C}} \cong \text{PSL}_2(\mathbb{C})$, but there are no "properly discontinuous subgroups" as all of these automorphisms have fixed points (blame $\hat{\mathbb{C}}$'s compactness). So the only option here is $M = \hat{\mathbb{C}}$.

The **Uniformization Theorem**, which we'll prove next week, essentially says that $\hat{\mathbb{C}}, \mathbb{C}, \mathbb{C}^*$, and $E_{\lambda}$ is the complete list of RS's whose universal cover is not D (equivalently, Tn). You may be familiar with the topology result that every compact orientable and 2-manifold, hence every compact RS, is a "sphere with handles attached":

![Diagram](image)

Uniformization asserts that any compact RS of genus $g \geq 2$
(or for that matter \( E^r \) or \( \hat{E}^r \)) is \( \tilde{\mathcal{D}} \) (or \( \tilde{\mathcal{G}} \)) for some \( \Gamma \subset \text{Aut}(D) \cong \text{Aut}(\hat{\mathbb{C}}) \cong \text{PSL}_2(\mathbb{R}) \). The Poincaré metric on \( D \) induces through this a metric of constant negative curvature on all but the 4 “exceptional” types of \( RS \) above!

Though others, including Köbe and Poincaré, made the proof more rigorous and natural, the idea of uniformization (and its proof) were due to Klein in 1882. The way he thought of it was that every \( RS \) \( M \) would be parametrized by a single complex variable defined on a subset of \( \hat{\mathbb{C}} \). That means you can write (say) all meromorphic functions on \( M \) in terms of this variable. We’ve already seen how powerful this is in the case of elliptic functions (defining functions on \( E \)) and modular functions (defining functions on modular curves like \( \Gamma(N)^{\infty} \)).

On Friday we’ll prove some technical results about Green’s functions on \( RS \), which will be needed in the proof of uniformization. (Another tool will be the Distortion Theorem of Lect. 32.) In the next short section, I set up (some of) the language.
II. Green's functions

Let $M$ be a RS, and let $V \subset \mathcal{H}(M)$ be a subset ("family"). (This time we will allow our subharmonic functions to tend to $-\infty$ at isolated points; but we will otherwise still take them to be continuous.) Note that the notion of subharmonicity is preserved under "precomposition" with holomorphic functions (viz., $\text{vof}$); otherwise the notion wouldn't make sense on RSs.

**Definition** $V$ is $\text{Perem} \iff \begin{cases} (i) \quad u, v \in V \Rightarrow \max(u, v) \in V \\ (ii) \quad u \in V, \Delta \subset M \text{ Jordan region} \\ \Rightarrow \min_{\Delta} u \in V. \end{cases}$

**Proposition** $V \text{ Perem} \Rightarrow u := \sup_{v \in V} u$ is either identically $+\infty$ or belongs to $\mathcal{H}(M)$.

**Proof:** We did this for plane regions; generalizes easy to RS. □

**Definition** Let $p_0 \in M$, $z$ a local coordinate about $p_0$, $V_{p_0} := \text{family of } u \in \mathcal{H}(M_{p_0})$ s.t. $\begin{cases} u \equiv 0 \text{ outside a compact set,} \\ \lim_{p \to p_0} \{u(p) + \log|z(p)|\} < \infty. \end{cases}$

† Last of Step 2 in the proof of Dirichlet (Chapter 10)
If \( \text{sup} \, \psi \) is finite, we say \( M \) has a Green's function
at \( p_0 \), denoted \( g(p, p_0) \).

Remark / (i) If a GF \( \psi \) at \( p_0 \), the range of \( \psi(p) \) contains
some \( D_{r_0} \). Set \( \psi_0(p) := \log r_0 - \log |\psi(p)| \) where
\( |\psi(p)| \leq r_0 \) and \( \psi_0(p) = 0 \) elsewhere \( \Rightarrow \psi_0 \in V_{p_0} \)
\( \Rightarrow g(p, p_0) \geq \psi_0(p) \Rightarrow \lim_{p \to p_0} g(p, p_0) = +\infty \)
(\( \Rightarrow \) nonconstant)

(ii) \( M \) compact \( \Rightarrow \) Green's fn. at any \( p_0 \).

(Otherwise, \( g(p, p_0) \) would have a minimum, hence
be constant)

(2ii) \( g(p, p_0) > 0 \) (since \( 0 \in V_{p_0} \), \( \geq 0 \) is clear;
and \( M \) open \( \Rightarrow \) \( g \) can't attain its minimum)

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III. Schlicht functions, concl.

There are a couple more interesting things to talk...
(A) MEANS

Theorem 1 (Littlewood, 1925) \( f \in l^2, \, r \in (0, 1) \Rightarrow \)

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})| \, d\theta \leq \frac{r}{1-r}.
\]

To prove this, we require a

Lemma: Given \( g \in l^2(D) \), set (for \( r \in (0, 1) \))

\[ I(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log(|re^{i\theta}|^2) \, d\theta. \]

Then \( r I'(r) = \frac{2}{2\pi} \int_{D_r} |g'(z)|^2 \, dA. \)

**Proof:**

\[ g(z) = \sum_{n \geq 0} b_n z^n \]
\[ g'(z) = \sum_{n \geq 1} nb_n z^{n-1} = \sum_{m \geq 1} \sum_{n \geq 0} b_n b_{n+m} z^n. \]

By Parseval,

\[ I(r) = \sum_{n \geq 0} |b_n|^2 r^{2n}, \quad r I'(r) = \sum_{n \geq 1} 2n |b_n|^2 r^{2n-2}, \]

and

\[
\frac{1}{2\pi} \int_{D_r} |g'(z)|^2 \, dA = \frac{1}{2\pi} \int_{0}^{r} R \, dR \int_{0}^{2\pi} |g'(re^{i\theta})|^2 \, d\theta
\]

\[ = \int_{0}^{r} R \sum_{n \geq 1} n^2 |b_n|^2 R^{2n-2} \, dR
\]
\[ = \sum_{n \geq 1} n^2 |b_n|^2 \int_{0}^{r} R^{2n-2} \, dR
\]
\[ = \sum_{n \geq 1} \frac{1}{2n} |b_n|^2 r^{2n-2} \]
\[ = \sum_{n \geq 1} \frac{1}{2n} |b_n|^2 r^{2n} = \frac{r}{1-r} I'(r). \]

\( \square \)
Proof of Thm. 1: Let \( f \in \mathcal{A} \), \( g(\theta) = \sqrt{f(e^{i\theta})} \) (\( \mathcal{A} \)).

We have
\[
I(r) = \frac{i}{2\pi} \int_{-\pi}^{\pi} |g(re^{i\theta})|^2 d\theta
= \frac{i}{2\pi} \int_{-\pi}^{\pi} |f(re^{2i\theta})| d\theta
= \frac{i}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})| d\theta.
\]

Now, \( g \) is \( 1-1 \); by the lemma,
\[
r^2 I'(r) = \frac{4}{2\pi} \int_{D_r} |g'|^2 dA = \frac{4}{2\pi} A(g(D_r))
\leq \frac{4}{2\pi} \pi \|g\|^2_{D_r} = 2 \|f\|^2_{D_r}.
\]
\[
\leq 2 \|K\|^2_{D_r^2} = \frac{2r^2}{(1-r^2)^2}.
\]

So
\[
\begin{align*}
I'(r) &\leq \frac{2r}{(1-r^2)^2} = \frac{d}{dr} \left( \frac{1}{1-r^2} \right) \\
I(0) &= |g(0)|^2 = 0, \quad \left. \frac{1}{1-r^2} \right|_0 = 1
\end{align*}
\]
\[
\Rightarrow I(r) \leq \frac{1}{1-r^2} - 1 = \frac{r^2}{1-r^2}.
\]
\[
\Rightarrow \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})| d\theta = I(\sqrt{r}) \leq \frac{r}{1-r}.
\]

Remark: \[
\frac{1}{2\pi} \int_{-\pi}^{\pi} |K(re^{i\theta})| d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{r}{1-re^{i\theta}} d\theta
\]
\[
\exp(-n) = r \sum_{n=0}^{\infty} r^{2n} = \frac{r}{1-r^2} = \frac{1}{1+r} \cdot \frac{r}{1-r} \sim \frac{1}{(r+1)} \frac{r}{1-r}.
\]
So Littlewood's estimate is asymptotically too high by a factor $\sqrt{2}$, in light of the following result:

**Theorem (Baernstein, 1974)**

$$f \in L^p, \quad p \in \mathbb{R}, \quad r \in (0, 1) \implies$$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^p d\theta \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |K(re^{i\theta})|^p d\theta.$$

Equality for any $r, p \implies f = \text{rotation of } K$.

**B) COEFFICIENTS**

As usual, we take $f = \sum_{n=0}^{\infty} a_n z^n \in \mathbb{S}$.

**Theorem 2 (Littlewood, 1925)**

$$|a_n| \leq e \cdot n, \quad n \geq 0.$$

**Proof:**

$$r^n a_n = \frac{1}{2\pi} \int_{0}^{2\pi} f(re^{i\theta}) e^{-in\theta} d\theta \quad \text{(by Cauchy)}$$

$$\implies r^n |a_n| \leq \frac{1}{2\pi} \int_{0}^{2\pi} |f(re^{i\theta})| d\theta \leq \frac{r}{1-r}.$$

Taking $r = 1 - \frac{1}{n}$, $(1 - \frac{1}{n}) |a_n| \leq n (1 - \frac{1}{n})$.

So $|a_n| \leq n (1 - \frac{1}{n})^{n-1} = \left(\frac{n}{n-1}\right)^{n-1} n < e n$

\[\begin{align*}
\text{take log + do L'Hospital:} \\
\lim_{x \to 0} \frac{\log \left(\frac{x}{x-1}\right)}{x} &= \lim_{x \to 0} \frac{1}{x(x-1)} = \lim_{x \to 0} \frac{x-1}{x} = 0 \\
\text{(and } \frac{x-1}{x} < 1 \text{)}
\end{align*}\]
We won’t prove the Bieberbach Conjecture / de Branges Theorem (i.e. the statement that \( |a_n| \leq n \), and that equality for some \( n \) \( \Rightarrow \) \( f \) is a rotation of \( K \)), but here is an easy special case:

**Theorem 3 (Dieudonné, 1931)**

Given \( f \in \mathcal{S} \) with all \( a_n \in \mathbb{R} \), we have \( |a_n| \leq n \) \( \forall n \geq 2 \).

**Proof:** First, observe that \( f(Da \mathbb{R}) \subset \mathbb{C} \) or \(-\mathbb{C}\): otherwise, since the image is connected, \( \exists a_0 \in D a \mathbb{R} \) s.t. \( f(a_0) \in \mathbb{R} \); and then \( \left| f_n \right| \subset \mathbb{R} \Rightarrow f \neq \frac{1}{1} \)

In fact, it is clear that \( f(Da \mathbb{R}) \subset \mathbb{C} \): just look at the form of \( f \sim f(x) = x + O(x^2) \).

Now let \( V := \text{Im}(f) \). Then \( V \big|_{D a \mathbb{R}} > 0 \), while

\[
V(re^{i\theta}) = \sum_{n \geq 1} a_n r^n \sin(n\theta) \quad \text{so for } n \geq 1
\]

\[
r^n a_n = \frac{2}{n} \int_0^n V(re^{i\theta}) \sin(n\theta) \, d\theta
\]

\[
\Rightarrow r^n |a_n| \leq \frac{2}{n} \int_0^n V(re^{i\theta}) \sin(n\theta) \, d\theta
\]
= n a \cdot r = n r.

Letting \( r \to 1^- \), we get \( |a_n| \leq n \).

Here is one more exotic variant. Recall ECG is starlike with resp. to 0 \( \iff \forall z \in E, \, t \in [0,1) \text{ we have } tz \in E \).

Set \( S^k := \{ f \in S \mid f(0) \text{ is starlike w.r.t. } 0 \} \).

**Theorem (Nevanlinna, 1920)** \( f \in S^k \Rightarrow |a_n| \leq n \) \((n \geq 2)\),

with the usual statement on "equality for some \( n \Rightarrow f \equiv c \cdot z^k \)."