Lecture 34: Green's functions on Riemann surfaces II

Let $M$ be a RS, and let $V \subset \overline{H}(M)$ be a subset ("family"). Recall:

**Definition** $V$ is **Peren** $\iff$

(i) $\forall_1, \forall_2 \in V \Rightarrow \max(\forall_1, \forall_2) \in V$

(ii) $\forall \in V$, $\Delta \subset M$ Jordan region $\Rightarrow \hat{\forall}_\Delta \in V$.

**Proposition 1** $V$ **Peren** $\Rightarrow u := \sup_{V} \forall$ is either identically $\infty$ or belongs to $\overline{H}(M)$.

**Definition** Let $p_0 \in M$, $t$ a local coordinate about $p_0$,

$V_{p_0} := \text{family of } \forall \in \overline{H}(M)(p_0)$ s.t. $\{ \forall \in 0 \text{ outside a compact set, }$

$s$ sup $\forall$ is finite, we say $\{ \sup_{p_0} \forall + \log \left| \forall(p) \right| \}_{p_0 \to p_0}$

$M$ has a **Green's function at** $p_0$, denoted $g(p, p_0)$.

If $M$ is compact, no GFs exist.

If $g(p, p_0)$ exists, it is $>0$ and limits to $\infty$ at $p_0$.

Now let $K \subset M$ be compact w/nonempty interior and $M \setminus K$ connected,

$V_K \subset \overline{H}(M \setminus K)$ the family s.t. $\{ \forall \in V_K \text{ is } \leq 1 \text{ (on } M \setminus K) \}

\text{Clearly this has } 0 \leq \forall_K \leq 1.$

Looking at the particular elements of $V_K$ defined by

$\forall(p) := \left\{ \begin{array}{ll} \log \left( 2 \frac{r}{|r(p)|} \right) / \text{comm} & \text{if } \exists \epsilon = \epsilon_0 \text{ and } \\
0 & \text{otherwise} \end{array} \right.$
we see $u_K \neq 0$, hence $u_K > 0$ (by the maximum principle for harmonic). Similarly if $u_K \neq 1$ then $u_K < 1$.

**Definition**

(a) If $u_K \neq 1$ the call it the harmonic measure of $K$ (i.e. this exists).

(b) Say the maximum principle is valid on $M\setminus K$ if, for any bounded above $u \in C(M\setminus K)$,

$$\lim_{p \to K} u(p) \leq 0 \implies u \leq 0 \text{ on } M\setminus K.$$  

**Proposition 2**

The following are equivalent:

(i) Green's functions exist ($\forall p_0 \in M$)

(ii) Harmonic measures exist ($\forall K \subset M$)

(iii) The maximum principle is NOT valid ($\forall M\setminus K$).

Moreover, $\exists$ of GF [resp. HM] for one $p_0$ [resp. one $K$] ensures their existence for all $p_0$ [resp. $K$].

**Proof:**

**STEP 1** ($\Rightarrow$)$p_0 \rightarrow (iii)_K$ if $p_0 \in K$:

$g(p, p_0)$ has a maximum $\mu$ on $K$, and is bounded above by 0 on $M\setminus K$. The maximum principle for $M\setminus K$ implies $g \leq \mu \implies \mu =$ maximum value for $-g$ on all of $M\setminus p_0$, assumed at some point of $K\setminus p_0 \implies g$ constant.
STEP 2 \([a]_K \Rightarrow (i)_{p_0}\) if \(p_0 \in \text{int}(K)\):

If \(u_K \neq 1\), then \(\sup_{v \in V_K} 1 \Rightarrow v_K \leq v\). So assume now that \(u_K\) exists, we have \(u_K\) too.

Now given \(v \in V_{p_0}\), we consider \(v^+ := \max (v, 0) \in V_{p_0}\). Notice that \(v^+(p) \leq (\max v^+) \cdot u_K(p)\) holds on \(\partial K_1\), and also near the "ideal boundary" \((p \to 00)\) (since \(v^+ \in V_{p_0}\) vanishes outside a compact set). So then this \(\leq \) must hold outside \(K_1\) by the usual maximum principle on \(M \setminus K_1\), and we have in particular

\[(\ast) \quad \max v^+ \leq (\max v^+) \cdot (\max u_K).\]

Next look at \(v^+(p) + (1+\varepsilon) \log |v(p)|\) on \(K_2\), which \(\to -\infty\) as \(p \to p_0\) (since \(\varepsilon > 0\)). It follows that

\[(\ast\ast) \quad \max v^+ + (1+\varepsilon) \log r_1 \leq \max v^+ + (1+\varepsilon) \log r_2.\]

Now \((\ast\ast) + (\ast) \Rightarrow\)

\(\Rightarrow \max v^+ \leq \max v^+ + (1+\varepsilon) \log (r_1/r_2) \leq (\max v^+) \cdot (\max u_K) + (\varepsilon) \cdot \log (r_1/r_2)\)

\(\Rightarrow \max v^+ \leq \frac{(1+\varepsilon) \log (r_1/r_2)}{1 - \max u_K}\)

\(\Rightarrow v^+, \text{ hence } v, \text{ is uniformly bounded above on } \partial K_1\)

\(\Rightarrow g(p, p_0) \text{ exists.}\)
STEP 3 (iii) $k \Rightarrow (ii)_k'$ \forall \( K, K' \): (This will finish the proof.)

We show \( L_k \Rightarrow L_k' \), so assume \( u_k' \neq 0 \), i.e., \( \sup_{v \in V_k'} \| v \| \equiv 1 \).

First assume \( K' \in K \). Let \( v \in V_{K'} \), and \( u \in \text{K}(M \setminus K) \) with \( u \leq 1 \) and \( \lim_{p \to K} u(p) \leq 0 \). Then \( v + u \leq 1 \) as \( p \to \infty \), \( p \to K \Rightarrow v + u \leq 1 \) on \( M \setminus K \) (max. principle). So taking \( v \) arbitrarily close to \( 1 \), we find \( u \leq 0 \). Hence the maximum principle is valid on \( M \setminus K \).

If \( K' \notin K \), choose \( K'' \) s.t. \( K \cup K' \subset \text{int}(K'') \). The assumed nonexistence of \( u_k' \Rightarrow MP \) holds for \( M \setminus K'' \) by the last paragraph. Let \( u \in \text{K}(M \setminus K) \) with \( u \leq 1 \), \( \lim_{p \to K} u(p) \leq 0 \). We want to show \( u \leq 0 \) (\( \Rightarrow MP \) for \( M \setminus K \)). First, by MP for \( M \setminus K'' \),

\[ u|_{M \setminus K''} \leq \max_{\partial K''} u. \]

Now suppose \( \max u > 0 \). By the usual max-min principle in \( K'' \setminus K \), and \( \lim_{p \to K} u \leq 0 \) (\( < \max u \)), we would have

\[ u|_{K'' \setminus K} \leq \max_{\partial K''} u. \]

By the 2 displayed steps, \( u \) attains its maximum at a point of \( \partial K'' \), which is interior to \( M \setminus K \) \( \Rightarrow u \) is constant. This is a contradiction, since it is \( > 0 \) somewhere on \( \partial K'' \) and \( \to 0 \) otherwise. Hence, \( \max u \leq 0 \), and by the usual maximum principle in \( K'' \setminus K \), \( u|_{K'' \setminus K} \leq 0 \). Also \( u|_{M \setminus K''} \leq 0 \), and so \( u \leq 0 \).
Corollary 1: If it exists, $g(p, p_0)$ satisfies the 3 properties:

(I) $g(p, p_0) > 0$

(II) $\inf g(p, p_0) = 0$

(III) $g(p, p_0) + \log |z(p)|$ has a harmonic extension to a nbhd. of $p_0$.

Proof: We already know (I). For (III), let $m(r) = \max_{|z(p)|=r} g(p, p_0)$.

(\textit{Note}) $m(r) + \log(r)$ is an increasing function of $r$

$\Rightarrow g(p, p_0) + \log |z(p)|$ is bounded above near $p_0$.

Also, $N(p) = \begin{cases} -\log |z(p)| + \log r_0, & |z(p)| < r_0, \quad p \in V_{p_0} \\ 0, & \text{otherwise} \end{cases}$

$\Rightarrow g(p, p_0) \geq -\log |z(p)| + \log r_0$

$\Rightarrow g(p, p_0) + \log |z(p)|$ bounded below near $p_0$.

Since isolated singularities of a bounded harmonic function are removable, done.

For (II), set $c = \inf g(p, p_0)$. By (III), $g(p, p_0) + \log |z(p)|$ has a finite limit as $p \to p_0$. So (by maximum)

$(1-e)N(p) \leq g(p, p_0) - c$ (\(\forall p \in V_{p_0}, \quad e > 0\))

\text{growth bad. as } p \to p_0 \text{ by } -(1-e)\log |z(p)|

Also, 0 outside compact set

$\Rightarrow (1-e)g(p, p_0) \leq g(p, p_0) - c$ (\(\forall e > 0\))

\text{def. of } g)

$\Rightarrow c \leq 0 \Rightarrow c = 0$. (def. of c; fact that $g > 0$)
Corollary 2: If $E$ bounded nonconstant $h \in X(M)$, then
if compact $K \subset M$ with nonempty interior, the maximum principle
on $MK$ fails, and $M$ has a Green's function w/ singularity
at any point $p \in M$.

Proof: $h$ has a maximum on $K$, attained (necessarily) at $K$.
were $NP$ for $MK$ valid, it would follow that this was
a maximum on all of $M$, rendering $h$ constant.

Now recall the Dirichlet principle. The proof generalizes
immediately to Riemann surfaces in the following sense:

Proposition 3: Let $M \subset \tilde{M}$ be RS. If $\partial M \subset \tilde{M}$
is a finite union of closed analytic arcs, then a bounded
$C^0$ func. $f : \partial M \to \mathbb{R}$, $\exists h : M \to \mathbb{R}$ s.t.

- $|h| \leq \text{sup} |f|$
- $h|_M \in X(M)$
- $h|_{\partial M} = f$

Corollary 3: Under the same hypothesis,
(i) $M$ has Green's function $g(p, p_0)$
(ii) $\lim_{p \to M} g(p, p_0) = 0$. 
Proof: (i) is an immediate consequence of Prop. 3 + Cor. 2.

For (iii), let $C$ be a small circle about $p_0$. Prop. 3 $\Rightarrow$

If solution $h$ to the Dirichlet problem outside $C$ w/boundary data $h|_C = f$, $h|_m = 0$. By the maximum principle for $\mathcal{X}$, $h$ is upper bound for $u \in \mathcal{V}_p$. $\Rightarrow \lim_{p \to p_0} g(p, p_0) \leq 0$

$\Rightarrow \lim_{p \to p_0} g(p, p_0) \geq 0$. 

$0$-function $cV_{p_0}$. 

$\square$