Lecture 36: Several complex variables I

Let \( D \subset \mathbb{C}^n \) be a domain (or more generally an open set).

**Definition** A function \( f : \Theta \rightarrow \mathbb{C} \) is holomorphic \( \iff \forall \alpha \in D \exists \text{ open } U \subset \Theta \text{ s.t.: } f \text{ has a power-series expansion } f(z) = \sum_{i=0}^{\infty} c_i (z - a)^i \text{ which converges } \forall z \in U. \)

In this lecture, which begins the final segment of the course, we shall only cover a few preliminaries, the most important of which is Osgood's lemma. (Finishing up Uniformization took some class time.)

**Notation:** \( \widetilde{D}(\alpha, \varepsilon) := \prod_{i=1}^{n} \widetilde{D}(\alpha_i, \varepsilon_i) \) polydisks

Clearly \( f \in \mathcal{H}(\Theta) \Rightarrow \text{ expansion converges absolutely/uniformly on each } U \text{ in the definition.} \)
\[\Rightarrow (i) \quad f \in C^0(\mathcal{B}), \text{ in particular } f \text{ is locally bounded (bounded on each compact subset of } \mathcal{B})\]

(ii) \(f\) is holomorphic in each variable separately (since it is a limit, fixing all but one variable, of holomorphic polynomials), i.e.,
\[
\overline{\partial}f := \frac{\partial f}{\partial \overline{z}_1} + \cdots + \frac{\partial f}{\partial \overline{z}_n} \, \overline{dz}_1 \wedge \cdots \wedge d\overline{z}_n = 0.
\]

In fact, the converse holds:

**Lemma 1 (Osgood)** \[f \text{ locally bounded, with } \overline{\partial}f = 0 \Rightarrow f \text{ holomorphic.}\]

**Proof:** Let \(\mathbf{z} = (z_1, \ldots, z_n) \in \mathcal{B},\)
\[
\overline{\Delta} := \overline{B}(z, r) \subset \mathcal{B}. \text{ Fix } (z_1, \ldots, z_n) \in \overline{\Delta}.
\]

\(f\) hol. in \(S_n \Rightarrow f(z_1, \ldots, z_{n-1}, z_n) = \frac{1}{2\pi i} \oint_{\partial S_n} \frac{f(s_1, \ldots, s_n, z_n)}{s_n - z_n} \, ds_n \quad (\text{each entry with the other entries fixed})
\]

\(f\) hol. in \(S_{n-1} \Rightarrow f(z_1, \ldots, z_{n-1}, z_n, z_{n-1}) = \frac{1}{2\pi i} \oint_{\partial S_{n-1}} \frac{f(s_1, \ldots, s_{n-1}, z_n)\, ds_{n-1}}{s_{n-1} - z_n}
\]

\[
\begin{align*}
&= \frac{1}{(2\pi i)^3} \oint_{\partial S_{n-1}} \left( \oint_{\partial S_n} \frac{f(s_1, \ldots, s_n, z_n)\, ds_n}{s_n - z_n} \right) \, ds_{n-1} \\
&\quad \text{Local boundedness allows us to rearrange the etc.}
\end{align*}
\]
resulting iterated integral to obtain

\[ f(a_1, \ldots, a_n) = \frac{1}{(2\pi)^n} \int \frac{f(s_1, \ldots, s_n)}{\prod (s_i - a_i)} \prod \, ds_i \]

Now

\[ \frac{1}{s_i - a_i} = \frac{1}{(s_i - d_i) - (a_i - d_i)} = \frac{1}{s_i - d_i} \cdot \frac{1}{1 - \frac{a_i - d_i}{s_i - d_i}} \sum_{j \geq 0} \frac{(a_i - d_i)^j}{(s_i - d_i)^{j+1}} \]

\[ \Rightarrow f(\xi) = \sum_{j \in \mathbb{Z}_{\geq 0}} \left( \frac{1}{(2\pi)^n} \int f(s_1, \ldots, s_n) \prod_{i=1}^n ds_i \right) \prod (\xi_i - a_i)^j \]

gives a power-series expansion at \( \xi \), necessarily convergent in \( \Delta \).

**Corollary 1 (Cauchy estimates)** Let \( \alpha = (\alpha_1, \ldots, \alpha_n) \) be a multi-index, and write \( |\alpha| = \sum \alpha_i \), \( \alpha! = \prod \alpha_i! \). In the above situation we have

\[ \alpha! Y_\alpha = \left\{ \frac{1}{(2\pi)^n} \int \frac{f(s_1, \ldots, s_n)}{\prod (s_i - a_i)^{\alpha_i+1}} \prod_{i=1}^n ds_i \right\} \]

with \( |Y_\alpha| \leq M/\gamma^\alpha \), where \( M = \sup_{\xi \in \Delta} |f(\xi)| \).

**Proof:** Differentiating both sides of (1) gives

\[ \left\{ \frac{1}{(2\pi)^n} \int \frac{f(s_1, \ldots, s_n)}{\prod (s_i - a_i)^{\alpha_i+1}} \prod_{i=1}^n ds_i \right\} \]

Setting \( z = \alpha \) leads immediately to both results.
Corollary 2 (Terman's Inequality) \[ \log |f(x)| \leq \frac{1}{\text{vol}(A)} \int_{A} \log |f(x)| \, dV(x) \]

**Proof:** Inserting the 1-variable inequality

\[ \log |g(x)| \leq \frac{1}{2\pi} \int_{0}^{2\pi} \log |g(x + re^{i\theta})| \, d\theta \]

gives

\[ \log |f(x)| \leq \frac{1}{(2\pi)^n} \int_{0}^{2\pi} \cdots \int_{0}^{2\pi} \log |f(x + r e^{i\theta})| \, d\theta \]

Take the product with \( \prod \rho_j \), then \( \int_{0}^{2\pi} \cdots \int_{0}^{2\pi} \rho \, d\rho \) to obtain

\[ \frac{1}{(2\pi)^2} \log |f(x)| \leq \frac{1}{2\pi} \int_{\mathbb{R}^2} \log |f(x)| \, dV(x) \]

hence the result.

\[ \square \]