Lecture 38: Several complex variables III

I. Bochner-Martinelli formula

Throughout, $\Omega \subset \mathbb{C}^n$ denotes a bounded domain with C^1 boundary: i.e. $\exists\, \rho \in C^1(U)$, $U \supset \partial\Omega$, s.t. $\Omega \cap U = \{\underline{z} \in U \mid \rho(\underline{z}) < 0\}$ and $\vec{\nabla}\rho \neq \vec{0}$ on $\partial\Omega$.

(This is equivalent to saying $\partial\Omega$ is a differentiable manifold.)

Let's start by recalling a version of Stokes's theorem.

A differential form of type (p,q) on Ω is a formal sum

$$\omega := \sum_{|\alpha|=p,\, |\beta|=q} \omega_{\alpha\beta}\, d\underline{z}_{\alpha} \wedge d\overline{\underline{z}}_{\beta} \quad ,$$

i.e. $dz_{\alpha_1} \wedge \cdots \wedge dz_{\alpha_p}$

and is said to be of class $C^k \Leftrightarrow$ the functions $\omega_{\alpha\beta}$ are.

We write $\omega \in E^{p,q}(C^k(\Omega))$, and

$$E^d(C^k(\Omega)) := \bigoplus_{p+q=d} E^{p,q}(C^k(\Omega)) \quad \left(\begin{array}{c}\text{differential forms}\\ \text{of degree } d\end{array}\right).$$

One has differential operators

$$\partial : E^{p,q}(C^k(\Omega)) \to E^{p+1,q}(C^{k-1}(\Omega))$$

$$\bar{\partial} : E^{p,q}(C^k(\Omega)) \to E^{p,q+1}(C^{k-1}(\Omega))$$

and $d = \partial + \bar{\partial} : E^d(C^k(\Omega)) \to E^{d+1}(C^{k-1}(\Omega))$, defined by $\partial\omega = \sum_{\alpha,\beta}\sum_{j} \frac{\partial \omega_{\alpha\beta}}{\partial z_j}\, dz_j \wedge d\underline{z}_\alpha \wedge d\underline{\bar z}_\beta$

$$\bar\partial\omega = \sum_{\alpha,\beta}\sum_{j} \frac{\partial \omega_{\alpha\beta}}{\partial \bar z_j}\, d\bar z_j \wedge d\underline{z}_\alpha \wedge d\underline{\bar z}_\beta .$$

Note that $\partial\circ\partial$, $\bar\partial\circ\bar\partial$, and $d\circ d$ are all 0. [Exercise]

The support of ω is the closure of the set in $\mathcal{D}$ where some $\omega_{\alpha\beta}$ is nonzero, $\mathrm{supp}(\omega)$. If this is compact we write $\omega \in E^{\cdots}(C^{\cdots}_c(\mathcal{D}))$. Finally, we have

$\boxed{\text{Stokes's Theorem}}$ For $\omega \in E^d(C^k(\bar{\mathcal{D}}))$, as usual, we mean "on an open nbhd. of" $\bar{\mathcal{D}}$

$$\int_{\partial\mathcal{D}} \omega = \int_{\mathcal{D}} d\omega .$$

Now we are ready to compute. We are after an analogue of the Cauchy integral formula (including the more general one for non-holomorphic functions) in several variables. Writing

$$\omega(\underline{z}) := dz_1 \wedge \dots \wedge dz_n = d\underline{z}$$

$$\eta(\underline{z}) := \left\langle \sum z_j \frac{\partial}{\partial z_j}, \omega(\underline{z}) \right\rangle = \sum (-1)^{j+1} z_j \overbrace{dz_1 \wedge \dots \wedge \widehat{dz_j} \wedge \dots \wedge dz_n}^{d\underline{z}\hat{\wedge}_j},$$

notice that Stokes $\Longrightarrow$

$$\int_{\partial B(\underline{z}^0,\epsilon)} \eta(\bar{\underline{z}}) \wedge \omega(\underline{z}) = \int_{B(\underline{z}^0,\epsilon)} \overbrace{n \cdot \omega(\bar{\underline{z}}) \wedge \omega(\underline{z})}^{(-1)^{\binom{n}{2}}(2i)^n\, dV}$$

$$= n\,(-1)^{\binom{n}{2}}(2i)^n\, \epsilon^{2n}\, \overbrace{\mathrm{Vol}\left(B^{2n}_{\mathbb{R}}\right)}^{\pi^n/n!}$$

$$= n\,\epsilon^{2n}\,(-1)^{\binom{n}{2}}\frac{(2\pi i)^n}{n!}$$

$$=: n\,\epsilon^{2n}\,W(n).$$

Now let $f \in C^1(\bar{\mathcal{D}})$, $\underline{z} \in \mathcal{D}$, $\mathcal{D}_{\underline{z},\epsilon} := \mathcal{D} \setminus \bar{B}(\underline{z},\epsilon)$,

$$L_{\underline{z}}(\underline{w}) := \frac{f(\underline{w})\,\eta(\bar{\underline{w}}-\bar{\underline{z}}) \wedge \omega(\underline{w})}{|\underline{w}-\underline{z}|^{2n}}.$$

Then

$$d_{\underline{w}}\left(L_{\underline{z}}(\underline{w})\right) = \frac{\bar\partial f(\underline{w}) \wedge \eta(\bar{\underline{w}}-\bar{\underline{z}}) \wedge \omega(\underline{w})}{|\underline{w}-\underline{z}|^{2n}}$$

$$\left[\text{Exercise: check that } \bar\partial\left(\frac{\eta(\bar{\underline{w}}-\bar{\underline{z}})}{|\underline{w}-\underline{z}|^{2n}}\right) = 0\right]$$

and

$$(*) \quad \int_{\mathcal{D}_{\underline{z},\epsilon}} \frac{\bar\partial f(\underline{w}) \wedge \eta(\bar{\underline{w}}-\bar{\underline{z}}) \wedge \omega(\underline{w})}{|\underline{w}-\underline{z}|^{2n}} \overset{\text{Stokes}}{=} \int_{\partial\mathcal{D}_{\underline{z},\epsilon}} L_{\underline{z}}(\underline{w})$$

$\left(O(|\underline{w}-\underline{z}|^{-2n+1}) \Rightarrow \text{integral converges as } \epsilon \to 0\right)$

$$= \int_{\partial\mathcal{D}} L_{\underline{z}}(\underline{w}) - \int_{\partial B(\underline{z},\epsilon)} L_{\underline{z}}(\underline{w})$$

where

$$\int_{\partial B(\underline{z},\epsilon)} L_{\underline{z}}(\underline{w}) = \underbrace{f(\underline{z})\int_{\partial B(\underline{z},\epsilon)} \frac{\eta(\bar{\underline{w}}-\bar{\underline{z}}) \wedge \omega(\underline{w})}{|\underline{w}-\underline{z}|^{2n}}}_{n\,\cancel{\epsilon^{2n}}\,W(n)/\cancel{\epsilon^{2n}}} + \underbrace{\int_{\partial B(\underline{z},\epsilon)} \frac{(f(\underline{w})-f(\underline{z}))\,\eta(\bar{\underline{w}}-\bar{\underline{z}}) \wedge \omega(\underline{w})}{|\underline{w}-\underline{z}|^{2n}}}_{|\cdot| \le \mathrm{vol}(\partial B)\cdot\frac{C\epsilon^2}{\epsilon^{2n}} = C'\epsilon \to 0}$$

hence

$$\lim_{\epsilon \to 0} (*) = \int_{\partial\mathcal{D}} L_{\underline{z}}(\underline{w}) - n\,W(n)\,f(\underline{z}).$$

So we have proved

$\boxed{\text{Theorem 1 (Bochner - Martinelli formula)}}$ Writing $C_n := \frac{1}{nW(n)}$,

$$f(\underline{z}) = C_n \int_{\partial D} \frac{f(\underline{w})\,\eta(\overline{\underline{w}}-\overline{\underline{z}})\wedge\omega(\underline{w})}{|\underline{w}-\underline{z}|^{2n}} - C_n \int_{D} \frac{\bar{\partial} f(\underline{w})\wedge\eta(\overline{\underline{w}}-\overline{\underline{z}})\wedge\omega(\underline{w})}{|\underline{w}-\underline{z}|^{2n}},$$

for any $f \in C^1(\bar{D})$ and $\underline{z} \in D$.

Remark// Note $C_n = \frac{(-1)^{\binom{n}{2}+1}(n-1)!}{(2\pi i)^n}$. The Bochner-Martinelli kernel is

$$K^n_{BM}(\underline{z},\underline{w}) = C_n \frac{\eta(\overline{\underline{w}}-\overline{\underline{z}})\wedge\omega(\underline{w})}{|\underline{w}-\underline{z}|^{2n}}$$

and is important not just here but in the theory of distributions/currents. The formula reads

$$f(\underline{z}) = -\int_{\partial D} f(\underline{w})\,K(\underline{z},\underline{w}) + \int_D \bar{\partial} f(\underline{w}) \wedge K(\underline{z},\underline{w}). \quad //$$

$\boxed{\text{Corollary 1}}$ (i) $[f \in C^1_c(D)]$ $f(\underline{z}) = \int_D \bar{\partial} f(\underline{w}) \wedge K(\underline{z},\underline{w})$

(special cases)

(ii) $[\bar{\partial} f = 0]$ $f(\underline{z}) = \int_{\partial D} f(\underline{w})\, K(\underline{z},\underline{w})$

(iii) $[n=1]$ $f(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(w)}{w-z}\,dw - \frac{1}{2\pi i}\int_D \frac{\partial f/\partial \bar{w}\,(w)}{w-z}\,d\bar{w}\wedge dw$

(iv) $[n=1\ \&\ \bar{\partial} f = 0]$ $f(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(w)}{w-z}\,dw$.

$\boxed{\text{Corollary 2}}$ $\bar{\partial} f = 0 \Rightarrow f$ is C^∞.

Proof: Apply Cor. 1 (ii) and differentiate under the $\int$, using the fact that (off of $\underline{w}=\underline{z}$) K is C^∞ in $\underline{w}$. $\square$

Remarks // (a) Contrast (ii) to $\displaystyle\frac{1}{2\pi i}\int_{\partial D_1\times\cdots\times\partial D_n}\frac{f(\underline{w})\,\omega(\underline{w})}{(w_1-z_1)\cdot\,\cdots\,\cdot(w_n-z_n)}$

— i.e. iterated Cauchy, on a subset of the boundary of a polycylinder. This "toric" nature makes it less flexible than the "spherical" B-M approach.

(b) In (ii), the kernel is, while $\bar\partial$-closed in z [Exercise], not holomorphic* (because the coefficients are not: e.g. for $n=2$, $\dfrac{(\bar w_1-\bar z_1)\,d\bar w_2-(\bar w_2-\bar z_2)\,d\bar w_1}{(|w_1-z_1|^2+|w_2-z_2|^2)^2}\wedge dw_1\wedge dw_2$). So you don't get a "representation theorem" for $n>1$ producing a holomorphic function from something C^1 or C^0 on ∂D. //

* This is possible b/c it is a differential form, not a function!

II. The $\bar\partial$-problem

Let $\psi \in C^1_c(\mathbb{C})$ — say, supported on some compact in D_R. We want to solve the 1-variable $\bar\partial$-problem

$$\bar\partial u(z) = \psi(z)\, d\bar z,$$

i.e. $\partial u/\partial \bar z = \psi$. By Corollary 1 (iii), together with compact support,

$$\psi(z) = \frac{-1}{2\pi i}\int_{D_R} \left(\frac{\partial\psi/\partial\bar w\,(w)}{w-z}\right) d\bar w \wedge dw$$

$$= \frac{-1}{2\pi i}\int_{\mathbb{C}} \frac{\partial\psi/\partial\bar w\,(w+z)}{w}\, d\bar w \wedge dw$$

change of coordinates

$$= \frac{-1}{2\pi i}\frac{\partial}{\partial\bar z}\int_{\mathbb{C}} \frac{\psi(w+z)}{w}\, d\bar w \wedge dw$$

$$= \frac{-1}{2\pi i}\frac{\partial}{\partial\bar z}\int_{\mathbb{C}} \frac{\psi(w)}{w-z}\, d\bar w \wedge dw,$$

and taking

$$u(z) := \frac{-1}{2\pi i}\int_{\mathbb{C}} \frac{\psi(w)}{w-z}\, d\bar w \wedge dw$$

does the job. So <u>convolution with the kernel $d\bar w \wedge dw / w$ solves the $\bar\partial$-problem in 1-variable.</u>

What about the several-variable case?

We start with $\psi = \sum \psi_j \, d\bar{z}_j \in E^{0,1}(C^1_c(\mathbb{C}^n))$.

If we are to have $\psi = \bar{\partial} u$, $u \in C^1(\mathbb{C}^n)$, then

$$\underset{\text{[Exercise]}}{0 = \bar{\partial}\bar{\partial}} \implies \bar{\partial}\psi = 0, \text{ i.e. } \boxed{\frac{\partial \psi_j}{\partial \bar{z}_\ell} = \frac{\partial \psi_\ell}{\partial \bar{z}_j} \ \forall j, \ell} \ (\dagger).$$

Theorem 2 (Poincaré lemma [for functions]) If $\bar{\partial}\psi = 0$, then for any $j \in \{1, \dots, n\}$

$$(**) \quad \boxed{u_j(z) := \frac{-1}{2\pi i} \int_{\mathbb{C}} \frac{\psi_j(z_1, \dots, z_{j-1}, \zeta, z_{j+1}, \dots, z_n)}{\zeta - z_j} \, d\bar{\zeta} \wedge d\zeta}$$

has $\bar{\partial} u_j = \psi$, and $u_j \in C^1_c(\mathbb{C}^n)$ (or $C^k_c(\mathbb{C}^n)$ if ψ is C^k).

Proof: $n = 1$ is done already. If $n > 1$, we claim that u_j is compactly supported (false for $n=1$!):

- $\partial u_j / \partial \bar{z}_\ell = \psi_\ell = 0$ for $|z|$ large $\implies$ u_j holo. in each variable separately $\underset{\text{Osgood}}{\implies}$ u_j holo. outside D^n_R
- $(**) \implies u_j = 0$ for $|\underline{z}_{\hat{j}}|$ large

$\implies u_j = 0$ outside D^n_R.

Now

Cor. 1 (iii) + compact support

$$\psi_\ell(\underline{z}) = \frac{-1}{2\pi i}\int_{D_R} \frac{\frac{\partial \psi_\ell}{\partial \bar z_j}(z_1,\dots,\zeta,\dots,z_n)}{\zeta - z_j}\, d\bar\zeta\wedge d\zeta$$

change of coords. + use (†)

$$= \frac{-1}{2\pi i}\int_{\mathbb{C}} \frac{\frac{\partial \psi_j}{\partial \bar z_\ell}(z_1,\dots,\zeta+z_j,\dots,z_n)}{\zeta}\, d\bar\zeta\wedge d\zeta$$

$$= \frac{-1}{2\pi i}\frac{\partial}{\partial \bar z_\ell}\int_{\mathbb{C}} \frac{\psi_j(z_1,\dots,\zeta+z_j,\dots,z_n)}{\zeta}\, d\bar\zeta\wedge d\zeta$$

$$= \frac{-1}{2\pi i}\frac{\partial}{\partial \bar z_\ell}\int_{\mathbb{C}} \frac{\psi_j(z_1,\dots,\zeta,\dots,z_n)}{\zeta - z_j}\, d\bar\zeta\wedge d\zeta. \qquad \square$$

Remark // The C^k part of the proof is an exercise using $\int$ by parts. All we'll need is ψ $C^\infty \Rightarrow u$ C^∞. //