Lecture 40: The Bergman kernel

Let $\mathcal{D} \subset \mathbb{C}^n$ be a domain, and consider the space of square-integrable holomorphic functions

$$A^2(\mathcal{D}) := \{ f \in \mathcal{H}(\mathcal{D}) \mid \left( \int_{\mathcal{D}} |f'(z)|^2 \, dV(z) \right)^{1/2} < \infty \}$$

with inner product

$$\langle f, g \rangle := \int_{\mathcal{D}} f(z) \overline{g(z)} \, dV(z), \quad f, g \in A^2(\mathcal{D}).$$

Last time we proved:

**Proposition** With this inner product, $A^2(\mathcal{D})$ is a Hilbert space.

**Key ingredient:** For $K \subset \mathcal{D}$ compact, $f \in A^2(\mathcal{D})$,

$$\sup_{z \in \mathcal{D}} |f(z)| \leq C_K \left( \int_{\mathcal{D}} |f(\zeta)|^2 \, dV(\zeta) \right)^{1/2}$$

**Theorem 1** There exists a unique function

$$K_\mathcal{D} : \mathcal{D} \times \mathcal{D} \to \mathbb{C} \quad (=: \text{Bergman kernel})$$

satisfying

(a) For each fixed $z \in \mathcal{D}$, $K_\mathcal{D}(z, \cdot) \in A^2(\mathcal{D})$ (as fcn of $\mathcal{D}$)

(b) $K(z, z) = 0$

(c) $f(z) = \int_{\mathcal{D}} K(z, \zeta) f(\zeta) \, dV(\zeta) \quad \forall f \in A^2(\mathcal{D})$. 

Proof: Observe that from (4) with \( K = \{ \pm \} \),

the "evaluation map" \( \Phi_\pm : A^2(\Omega) \to \mathbb{C} \)

\[ f \mapsto f(\pm) \]

is a continuous linear functional, i.e. \( \in A^2(\Omega)^* \). By

first, \( \exists \ k_\pm \in A^2(\Omega) \) s.t. \( \Phi_\pm(f) = \langle f, k_\pm \rangle \) \( \forall f \in A^2(\Omega) \),

i.e.

\[ f(\pm) = \langle f, k_\pm \rangle. \]

Set \( K(\pm, \xi) := \frac{k_\pm(\xi)}{k_\pm(\pm)}. \) Then

\[ \int_\Omega K(\xi, \xi) f(\xi) \, d\nu(\xi) = \langle f, k_\pm \rangle = f(\pm) \Rightarrow (c1). \]

Further,

\[ \int_\Omega K(\xi, \xi) K(\xi, \xi) \, d\nu(\xi) = \frac{1}{K(\xi, \xi)} \]

\[ \int K(\xi, \xi) K(\xi, \xi) \, d\nu(\xi) = \frac{1}{K(\xi, \xi)} = K(\xi, \xi) \]

\[ \Rightarrow (b). \] Finally, (a) now follows from \( K(\pm, \xi) = \overline{K(\xi, \xi)} = k_\pm(\xi). \)

To see uniqueness, let \( K'(\xi, \xi) \) be another such function. Then

\[ K(\pm, \xi) = \overline{K(\xi, \xi)} = \int K'(\xi, \xi) K(\xi, \xi) \, d\nu(\xi) \]

\[ = \int K(\xi, \xi) K'(\xi, \xi) \, d\nu(\xi) \]

\[ = \frac{1}{K'(\xi, \xi)} = K'(\pm, \xi). \]
Theorem 2. The Bergman kernel has the invariance property with respect to biholomorphic maps $f: \mathcal{D}_1 \to \mathcal{D}_2$

$$(\det J_f(z))(\det \overline{J_f}(\overline{z})) K_{\mathcal{D}_2}(f(z), f(\overline{z})) = K_{\mathcal{D}_1}(z, \overline{z}).$$

Proof: Given $\phi \in A^2(\mathcal{D}_1)$, by uniqueness of $K_{\mathcal{D}}$, the following calculation will suffice:

$$\int_{\mathcal{D}_1} (\det J_f(z))(\det \overline{J_f}(\overline{z})) K_{\mathcal{D}_2}(f(z), f(\overline{z})) \phi(z) \, dV(z) =$$

$$\int_{\mathcal{D}_2} (\det J_f(z))(\det \overline{J_f}(\overline{z})) K_{\mathcal{D}_2}(f(z), f(\overline{z})) \phi(f^{-1}(z)) \frac{1}{(\det J_f(f^{-1}(z)))^2} \, dV(z') =$$

$$\int_{\mathcal{D}_2} K_{\mathcal{D}_2}(f(z), f(\overline{z})) \phi(f^{-1}(z)) \, dV(z') = \frac{1}{(\det J_f(f^{-1}(z)))^2} \frac{1}{(\det J_f(f^{-1}(z)))^2} \, dV(z') = \phi(z).$$

Theorem 3. Let $\{\phi_j\}_{j=1}^\infty \subset A^2(\mathcal{D})$ be a complete orthonormal basis. Then

$$K_{\mathcal{D}}(z, \overline{z}) = \sum_{j=1}^\infty \overline{\phi_j(z)} \phi_j(\overline{z}).$$

† exists if

$\{\text{separability of } L^2(\mathcal{D})\}$

$\{ L^2(\mathcal{D}) = A^2(\mathcal{D}) \}$

$\Rightarrow$ separability of $A^2(\mathcal{D})$. 
Prof: The 3 properties are essentially clear: think of (c) or Fourier-expanding \( f \in A^2(B) \). All that needs checking is uniform convergence on \( K \times K \) (for \( K \subset B \) compact). The idea is

\[
\sup_{x \in K} \left( \sum_{j=1}^{\infty} |\varphi_j(x)|^2 \right)^{1/2} = \sup_{x \in K} \left| \sum_{j=1}^{\infty} \varphi_j(x) \right|^{1/2} \quad \text{for sequence}\n\]

\[
= \sup_{x \in K} \left| \sum_{j=1}^{\infty} \varphi_j(x) \right| \quad \text{(by (**) )}
\]

\[
= \sup_{\|f\|_{A^2(B)}=1} \|f\|_K \leq C_K \quad \text{(by (**))}
\]

\[
\Rightarrow \sum_{j=1}^{\infty} |\varphi_j(x)\varphi_j(y)| \leq \sum_{j=1}^{\infty} |\varphi_j(x)|^2 \cdot \sum_{j=1}^{\infty} |\varphi_j(y)|^2 \cdot \frac{1}{k^2}
\]

\[\text{and the convergence is uniform over } x, y \in K.\]

Example: \( \Theta = B^n \). The functions

\[\left\{ \varphi_j(x) \right\}_j \text{ multilinear } \subset A^2(B^n)\]
we a complete orthogonal system, since

- we can write any \( f \in A^2 \) as a power series uniquely
- \( \int_{B^n} z^a \overline{z}^b \, dV = 0 \) unless \( a = b \) (this boils down to \( \int_{S^p} e^{i\lambda} e^{i\theta} d\theta = 0 \) unless \( a = b \)).

Further, writing \( Y_a := \int_{B^n} |z^a|^2 \, dV(z) \), we have that \( \{Y_a\}_{a \geq 0} \) yield an orthonormal system and

- \( K(z, z') = \sum_{a \geq 0} Y_a \overline{z}^a z'^a \) (using Thm. 3)
- \( \mathcal{Y}_a^{(k)} = \frac{n^k k!}{(n+k)!} \) \( \square \) (Exercize)

where \( \mathbf{1} = (1, 0, \ldots, 0) \in \mathbb{C}^n \), and \( 0 < r < 1 \).

Then
\[
K_{B^n}(z, r \mathbf{1}) = \sum_{a \geq 0} \frac{z^a \overline{(r \mathbf{1})}^a}{Y_a} = \sum_{k \geq 0} \frac{z^k \overline{r}^k}{\pi^n k!} (n+k)!
\]

\[
= \frac{n!}{\pi^n} \sum_{k \geq 0} \binom{k}{r_1 \ldots r_m} \binom{k}{r_1 \ldots r_m} \binom{k}{r_1 \ldots r_m} \binom{k}{r_1 \ldots r_m}
\]

\[
= \frac{n!}{\pi^n} \cdot \frac{1}{(1-\overline{r})^{n+1}}. \quad (k\star)
\]

More generally, if \( \overline{z} = r \overline{z} \in B^n \) where \( \|\overline{z}\| = 1 \) and \( \{\rho \in U(n) \mid \rho(\overline{z}) = 1\} \)
$K_0(z, s) = K_0(r \tilde{z}, s) = K(r \tilde{z}, s) = K(r \tilde{z}, \rho(s))$

Then

$$K_0(z, s) = K_0(r \tilde{z}, s) = K(r \tilde{z}, s) = K(r \tilde{z}, \rho(s))$$

$$= \frac{n!}{\pi^n} \cdot \frac{1}{(1 - r \tilde{z} \cdot \rho(s))^n}$$

= \frac{n!}{\pi^n} \cdot \frac{1}{(1 - |r \tilde{z} \cdot \rho(s)|)^n}

= \frac{n!}{\pi^n} \cdot \frac{1}{(1 - 2 \tilde{z} \cdot \tilde{s})^n}

= \frac{n!}{\pi^n} \cdot \frac{1}{(1 - 2 \tilde{z} \cdot \tilde{s})^n}

For the unit disk ($n=1$), this is

$$K_0(z, s) = \frac{1}{\pi} \cdot \frac{1}{(1 - z \cdot \tilde{s})^2}.$$ 

By the invariance result (Theorem 2), if

$$f : \mathbb{D} \rightarrow \mathbb{D}$$

is a mapping as in the RMT, then

$$K_0(z, s) = \frac{f'(z) f'(s)}{\pi} \cdot \frac{1}{(1 - f(z) \overline{f(s)})^2}.$$
In a more thorough discussion we would have treated the Bergman metric.

**Theorem 4** Every bounded domain has a canonical Hermitian metric (i.e., Bergman metric). It has negative curvature and is invariant (pulls back to itself) under holomorphic automorphisms.

**Proof (partial sketch):** \( h_G := \int \int \log k_G(z, \bar{z}) \)
\[ = \sum_{i, j} \frac{2}{2i \pi j} \log |k_{0i0j}(z)| \, dz \, d\bar{z} , \]
which makes sense if \( k_{0i0j}(z) = \sum_{j} |\phi_j(z)|^2 > 0 \)
if \( = 0 \) anywhere then \( \exists \) a point \( z \) s.t. every \( f \in L^2(B) \)
various term -- absurd.

Meaning of this is that for \( \gamma : [0, 1] \to B \),
\[ \text{length}(\gamma) := \int_0^1 \sqrt{\sum_{i, j} h_{ij}(\gamma(t)) \gamma_i^2(t) \gamma_j^2(t)} \, dt . \]
To each \( z \) and 2-plane \( P = T_z B \), one assigns a number \( K(P) \) = Gaussian curvature. For the "holomorphic 2-planes" this number is \( < 0 \). (We won't have time
to prove this.) Invariance under conformal isomorphisms (of \( \Omega \) with itself or with another domain) is a direct consequence of Theorem 2.

\[ \begin{align*}
\frac{d}{dz} \frac{d}{d\overline{z}} \log \frac{1}{(1-|z|^2)^2} &= -2 \frac{2}{dz} \frac{2}{d\overline{z}} \log (1-|z|^2) \\
&= 2 \frac{1}{dz} \frac{z}{1-|z|^2} \\
&= \frac{2}{(-1)(1)^2} [dz \, d\overline{z}, \text{i.e. } dx^2 + dy^2] \\
\end{align*} \]

is the \text{Poincaré metric}. It "pulls back" to \( \frac{1}{4y^2} (dx^2 + dy^2) \) on the upper half-plane (can you show this?)