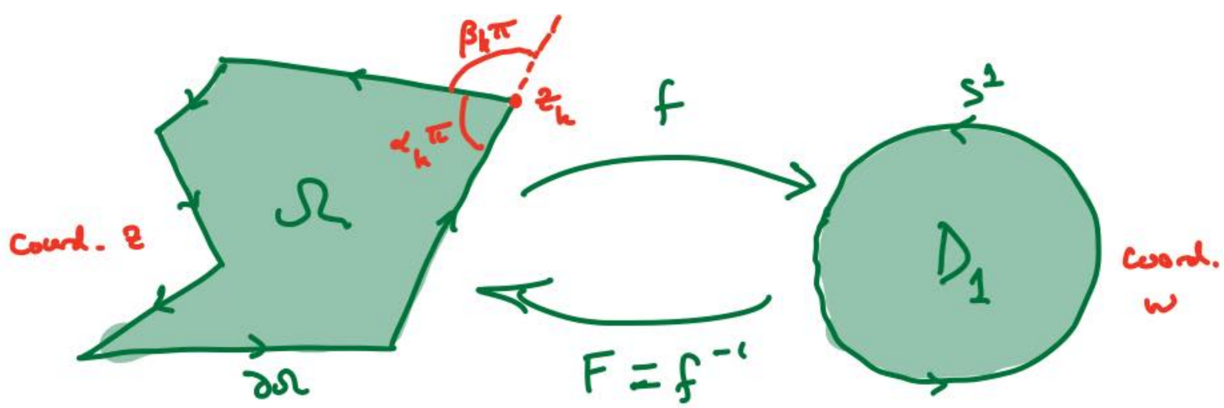
Lecture 5: Explicit conformal mappings

I. "Explicit RMT" for polygons

We would like to make reasonably concrete the map guaranteed by RMT (Lect. 2–4) — i.e. a conformal equivalence $\Omega \xrightarrow{\cong} D_1$ with C^0 extension to $\overline{\Omega} \xrightarrow{\cong} \overline{D}_1$ — between an n-gon and the unit disk:

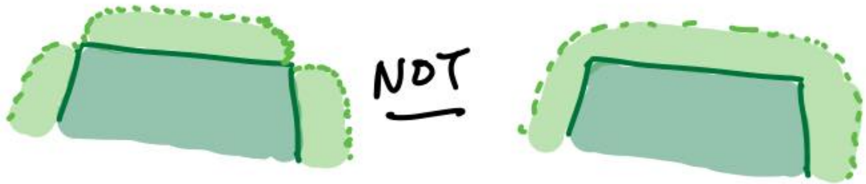


Here the n-gon is specified, up to a shift, by the data $\{\alpha_k\}_{k=1}^n = \{1-\beta_k\}_{k=1}^n$ where k indexes the vertices counterclockwise and

$$\sum_{k=1}^n \beta_k = 2 \ , \quad \beta_k \in (-1,1)$$

$$\left(\leftrightarrow \sum_{k=1}^n \alpha_k = \sum (1-\beta_k) = n - \sum \beta_k = n-2 \ , \quad \alpha_k \in (0,2) \right).$$

Schwarz reflection provides extensions to the neighborhoods of the segments :

i.e. 

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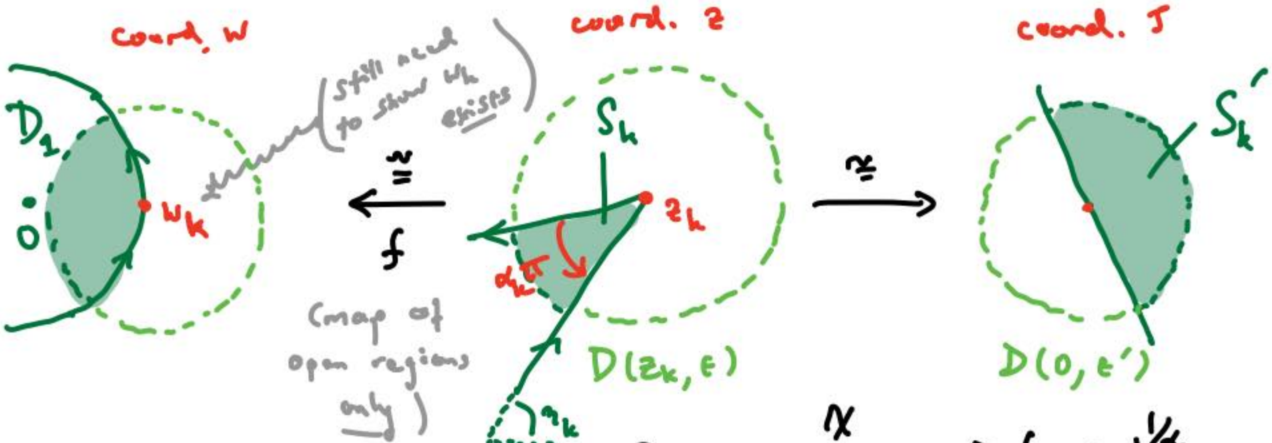
To get an explicit formula, it will actually help to start by explaining from the beginning how we get the C^0 extension to the whole boundary (including vertices), using only Schwarz reflection (not Carathéodory's Theorem).

Consider the picture



$$z \xmapsto{\ \chi\ } (z - z_k)^{1/\alpha_k} = \zeta$$

$$\begin{tikzcd}
f(z_k + \zeta^{\alpha_k}) & z_k + \zeta^{\alpha_k} \arrow[l, mapsto, "f"'] & \zeta \arrow[l, mapsto, "\chi^{-1}"'] \arrow[ll, bend left, "g"]
\end{tikzcd}$$

$$\begin{matrix} f(z_k + \zeta^{\alpha_k}) \\ \| \\ g(\zeta) \end{matrix}$$

where f is guaranteed by RMT. We know that

$$\zeta \to \text{diameter} \subset \partial S_k' \implies g(\zeta) \to \partial D_1 = S^1$$
$$\implies |g(\zeta)| \to 1.$$

So as in Lecture 3, applying Schwarz reflection to $\log(g(\zeta))$ yields an extension $\tilde{g}$ of g to $D(0,\epsilon')$. The same argument shows that $\arg(\tilde{g}(\zeta))$ increases along the diameter, hence we get a 1-to-1 C^0 extension of f to $\overline{S}_k$ by $\tilde{f}|_{\overline{S}_k} := \tilde{g}|_{\overline{S}_k'} \circ \chi|_{\overline{S}_k}$. One has to be careful because there is <u>not</u> an extension of f to $D(z_k,\epsilon)$. Set $w_k := \tilde{f}(z_k)$.

Now $\tilde{g}$ is holomorphic and 1-to-1 on $D(0,\epsilon')$, so we get

$$(w=)\ f(z_k+\zeta^{\alpha_k}) = g(\zeta) = w_k + \sum_{m\geq 1} \alpha_m \zeta^m, \quad \alpha_1 \neq 0$$

$$\implies (\zeta=)\ g^{-1}(w) = \sum_{m\geq 1} \beta_m (w-w_k)^m \quad \text{on some disk about } w_k, \quad \beta_1 \neq 0$$

'inverse fcn. thm.'

$$g^{-1}(w) = (\chi\circ F)(w) = (F(w)-z_k)^{1/\alpha_k}$$

$$\implies F(w) - z_k = \Big(\sum_{m\geq 1} \beta_m (w-w_k)^m\Big)^{\alpha_k}$$

raise to α_k

$$= (w-w_k)^{\alpha_k} \underbrace{\Big\{\overset{\overset{0}{\neq}}{\beta_1} + \sum_{m\geq 2} \beta_m (w-w_k)^{m-1}\Big\}^{\alpha_k}}_{=: G_k(w) \left(\text{holo. on disk about } w_k \text{ via binomial series}\right)}$$

$$\underset{\text{differentiate}}{\Longrightarrow} \quad F'(w) = \alpha_k (w-w_k)^{\alpha_k - 1} G_k(w) + (w-w_k)^{\alpha_k} G_k'(w)$$

$$\underset{\beta_k = 1-\alpha_k}{\Longrightarrow} \quad F'(w)\,(w-w_k)^{\beta_k} = \alpha_k \underbrace{G_k(w)}_{\text{nonvanishing at } w_k} + (w-w_k)\, G_k'(w).$$

By hypothesis (from RMT), $F'(w)$ is nonvanishing on D_1

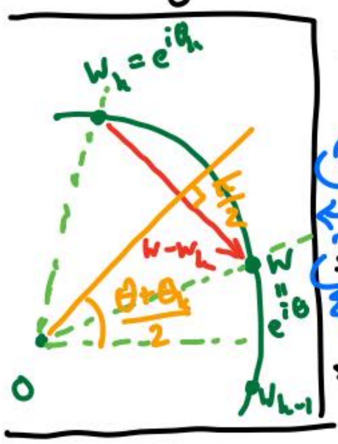
$$\Longrightarrow \quad F'(w) \cdot \prod_{k=1}^{n} (w-w_k)^{\beta_k} =: H(w)$$

has an extension to a nonvanishing holomorphic function on an open neighborhood of $\overline{D_1}$.

On S^1 between w_{k-1} and w_k, writing $w = e^{i\theta}$ and $w_k = e^{i\theta_k}$,

$$\sum \beta_k \arg(w-w_k) = \sum \beta_k \left(\frac{\theta+\theta_k}{2} - \frac{\pi}{2}\right) = \theta + \frac{\sum \beta_k \theta_k}{2} - \pi$$

$$\arg H(w) = \arg F'(w) + \sum_{k=1}^{n} \arg\left((w-w_k)^{\beta_k}\right)$$



$$= \arg\left(\frac{d}{d\theta} F(e^{i\theta})\right) - \arg\left(\frac{d}{d\theta} e^{i\theta}\right) + \sum \beta_k \arg(w-w_k)$$

$$= \eta_k - \left(\theta + \frac{\pi}{2}\right) + \sum \beta_k \left(\frac{\theta+\theta_k}{2} - \frac{\pi}{2}\right)$$

$$\underset{\sum \beta_i = 2}{=} \eta_k - \cancel{\theta} - \frac{\pi}{2} + \cancel{\theta} + \frac{1}{2}\sum \beta_k \theta_k - \pi$$

$$= \eta_k + \frac{1}{2}\sum \beta_k \theta_k - \frac{3\pi}{2},$$ which is constant.

$$\underset{H \text{ is } C^0}{\Longrightarrow} \quad \arg H(w) \text{ is constant on } S^1 \text{ (and harmonic in } D_1)$$

$$\underset{\text{Poisson formula or maximum principle}}{\Longrightarrow} \quad \arg H(w) \text{ is constant on } \overline{D_1}$$

$$\Longrightarrow \quad H(w) \text{ constant on } \overline{D_1}$$

$$\Longrightarrow \quad F'(w) = \frac{C}{\prod_{k=1}^{n} (w - w_k)^{\beta_k}},$$

proving the

$\boxed{\text{Theorem (Schwarz}^{1869}\text{-Christoffel}^{1867}\text{)}}$ The RMT-guaranteed conformal isomorphism $F : D_1 \to \Omega$ (polygon) takes the form

$$F(w) = C \int_0^w \frac{dw'}{\prod_{k=1}^{n} (w' - w_k)^{\beta_k}} + C'.$$

Remark// If you want to actually find the mapping, you need to know the $\{w_k\}$. We can always choose $w_n = 1$, $w_1 = -1$, $w_2 = -i$ (say) since the FLT taking an arbitrary w_1, w_2, w_n (arranged counter-clockwise) to those points is an automorphism of D_1. So for triangles you only need the $\{\beta_k\}$. As we'll see, already for the rectangle, the correspondence between w_3 (once w_1, w_2, w_4 are fixed as above) and the side-length data for Ω is highly transcendental! //

Now composing F with

$$g : h \longrightarrow D_1$$

$$\left(\begin{array}{c} \zeta \longmapsto \dfrac{\zeta - i}{\zeta + i} \\ i\,\dfrac{1+w}{1-w} \longleftarrow\!\shortmid\; w \end{array} \right)$$

should give a Schwarz-Christoffel formula for $h \xrightarrow{G} \Omega$:

Compute

$$g^* \left(\frac{dw}{\prod_{k=1}^n (w - w_k)^{\beta_k}} \right) = \frac{d(g(\zeta))}{\prod_k (g(\zeta) - w_k)^{\beta_k}} = \frac{\frac{2i}{\cancel{(\zeta+i)^2}}\, d\zeta}{\cancel{(\zeta+i)^{-2}} \prod_k (\zeta - i - w_k(\zeta+i))^{\beta_k}}$$

$$= \frac{2i\, d\zeta}{\prod_k \left(\zeta - i\,\frac{1+w_k}{1-w_k}\right)^{\beta_k} (1-w_k)^{\beta_k}} = \frac{2i}{\prod_k (1-w_k)^{\beta_k}} \cdot \frac{d\zeta}{\prod_k \big(\zeta - \underbrace{g^{-1}(w_k)}_{=:\, \zeta_k}\big)^{\beta_k}}$$

so that

$$G(\zeta) = K \cdot \int_0 \prod_{k=1}^n (\zeta - \zeta_k)^{-\beta_k} d\zeta \; + K' ,$$

<u>unless</u> (as in the Remark) some w_i (say, w_n) is 1.

<u>Then</u> we have a $(\zeta - i - (\zeta + i))^{\beta_n} = (-2i)^{\beta_n}$ in the denominator, so the final form just has a $\prod_{k=1}^{n-1}$,

and "$\zeta_n = \infty$".

II. The rectangle and elliptic functions

So let's turn this around: suppose we start with

$$\int_0^w \frac{dw'}{\sqrt{\Pi_{k=1}^4 (w'-w_k)}} =: F(w) \in C^0(\overline{D}_1) \cap \mathcal{H}ol(D_1)$$

(all $\beta_i = \frac{1}{2} = \alpha_i$), and ask what the image $F(D_1)$ looks like. (It's not *a priori* clear that the converse of the Theorem is true — i.e. that $F(D_1)$ is a polygon.) Now Lecture 4 Thm. A implies that <u>if we can prove F maps ∂D_1 into the boundary of a rectangle ∂R</u>, then $F(D_1) \subset R$ (hence is disjoint from $F(\partial D_1)$). Moreover, while F isn't defined on a neighborhood of $\overline{D}_1$, one can extend the proof of Lect. 4 Thm. B to our setting ($\gamma = \partial D_1$, F instead of f, etc.) — just take a limit in the argument-principle calculation in the Proof.

We conclude:

$$F \circ S^1 = \underset{\substack{\text{(traversed once}\\ \text{counterclockwise)}}}{\partial R} \implies F : D_1 \xrightarrow{\cong} R .$$

Since $G(\xi)$ is just the pullback of $F(w)$, we have

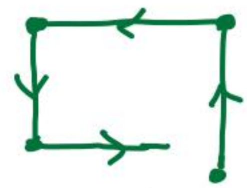
$$(*)\quad \boxed{G\circ \underbrace{(\mathbb{R}\cup\{\infty\})}_{\substack{\text{viewed as a}\\ \text{path } (=\partial h)}} = \partial R \implies G: h \xrightarrow{\cong} R\,.}$$

Now $G(\xi) = \displaystyle\int_0^{\xi} \frac{d\xi'}{\sqrt{\xi'(\xi'-1)(\xi'-\rho)}}$ (only 3 factors (assume $w_4 = 1$)), and if we take the branches of $\sqrt{\xi}$, $\sqrt{\xi-1}$, $\sqrt{\xi-\rho}$ sending

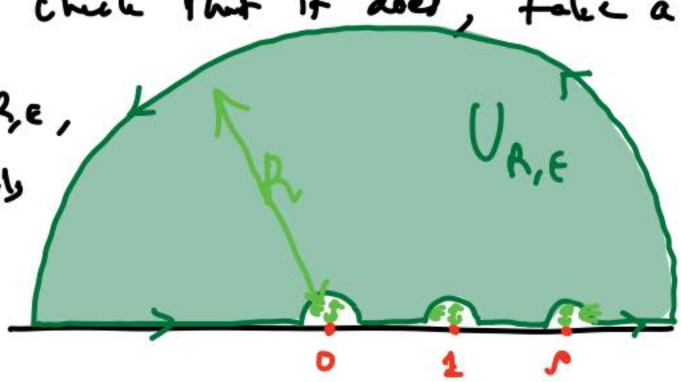
so that

- on $(-\infty, 0)$ we are integrating $\frac{1}{i^3}$ times a positive real function
- on $(0, 1)$ . . . . $\frac{1}{i^2}$ . . . .
- on $(1, \rho)$ . . . . $\frac{1}{i}$ . . . .
- on (ρ, ∞) . . . . 1 . . . .

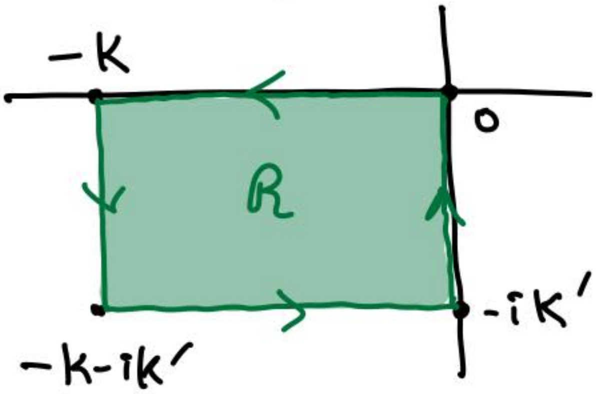
$\implies$ $G(\mathbb{R}\cup\infty)$ takes the form



Does it close up? To check that it does, take a limit of integrals over $\partial U_{R,\epsilon}$, where the semicircular integrals go to 0 as $\sim \frac{1}{\sqrt{R}}$ resp. $\sim \sqrt{\epsilon}$. Since there are



no residues of $\dfrac{d\xi}{\sqrt{\xi(\xi-1)(\xi-\rho)}}$ in $U_{R,\epsilon}$, the $\int_{-\infty}^{\infty}$ is <u>zero</u> and the rectangle closes up:



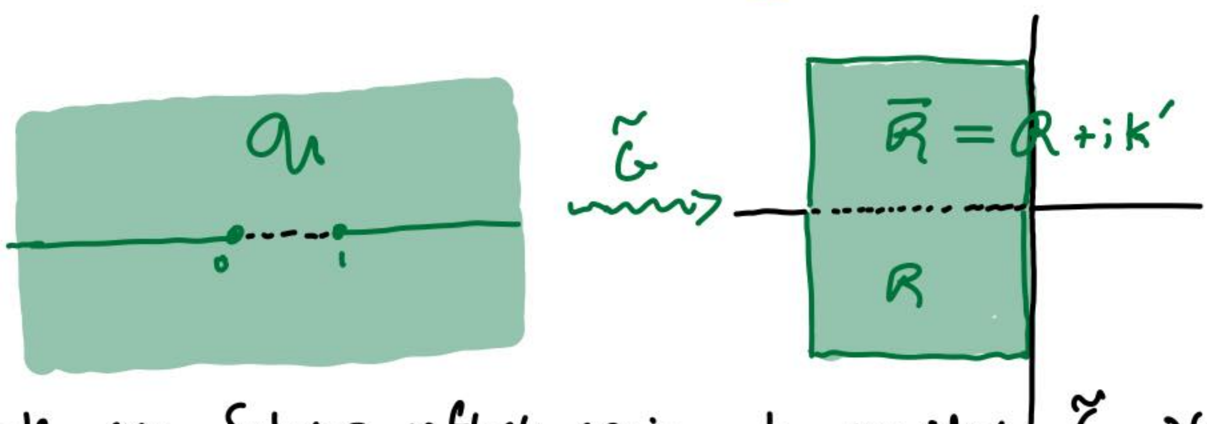
where $K := \int_0^1 \frac{dt}{\sqrt{t(1-t)(\rho-t)}}$ and $K' := \int_1^\rho \frac{dt}{\sqrt{t(t-1)(\rho-t)}}$.

Conclude that (via (*)) $G : \mathfrak{h} \xrightarrow{\cong} R$.

Now let $\mathcal{U} := \mathbb{C} \setminus (-\infty, 0] \cup [1, \infty)$ (symmetric about $\mathbb{R}$) and perform Schwarz reflection ($\mathcal{U} \cap \mathbb{R} = (0,1) \to (-K, 0) \subset \partial R$) to extend to

cx. conjugate region

$$\tilde{G} : \mathcal{U} \to R \cup \overline{R} \cup (-K, 0)$$



We can Schwarz-reflect again, by considering $\tilde{G} - iK'$

and reflecting through (p, ∞), then adding back iK'.

That is:

$$\underbrace{G(\xi)}_{h \to \mathbb{R}} \longmapsto \overline{\underbrace{G(\bar{\xi})}_{-h \to \mathbb{R}+iK'}} \longmapsto \overbrace{\overline{G(\bar{\xi})} - iK'}^{-h \to \mathbb{R}} \longmapsto \overbrace{\overline{G(\bar{\bar{\xi}})} - iK'}^{h \to \mathbb{R}+iK'}$$

$$\underbrace{G(\xi) + 2iK'}_{h \to \mathbb{R}+2iK'} \longleftarrow G(\xi) + iK' = \overline{G(\bar{\bar{\xi}})} - iK'$$

What this says, is that the <u>inverse function</u> G^{-1} extends to a mapping (still holomorphic) on $\overline{\mathbb{R} \cup (\mathbb{R}+iK') \cup (\mathbb{R}+2iK')}$ which has <u>period</u> $2iK'$. Repeating the process in the horizontal & vertical directions shows that G^{-1} extends to

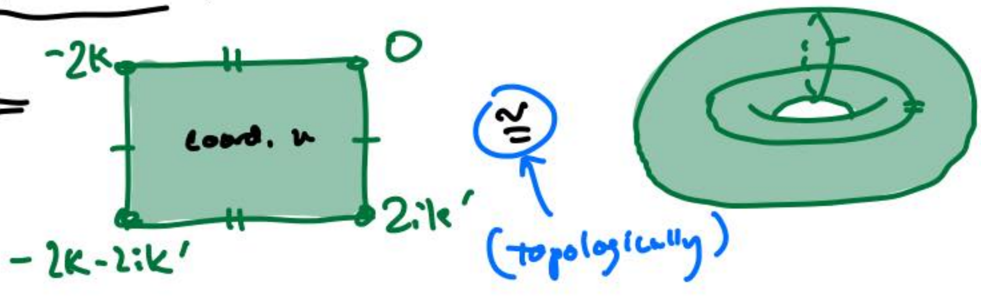
$$\wp : \mathbb{C} \setminus \Lambda \longrightarrow \mathbb{C} \setminus \{0, 1, p\}$$

$$(\text{or } \mathbb{C} \to \mathbb{C} \cup \{\infty\} = \mathbb{P}^1)$$

where $\Lambda = \mathbb{Z}\langle K, iK' \rangle$ and $\wp(u + 2Km + 2iK'n) = \wp(u)$.

That is, we get a <u>doubly-periodic function</u> called the <u>Weierstraß $\wp$-function</u>, well-defined on (i.e. <u>factoring through</u>)

$\mathbb{C}/2\Lambda =$



(Together with its derivative $\wp'$, it embeds $\mathbb{C}/\Lambda$

in projective 2-space with equation $y^2 = x(x-1)(x-\rho)$.)

It's important to note here that while the differential

$$\text{"}du\text{"} = \frac{d\xi}{\sqrt{\xi(\xi-1)(\xi-\rho)}}$$ is well-defined on "2 copies of $\mathbb{C}\setminus\{0,1,\rho\}$",

or really on the torus† $\mathbb{C}/2\Lambda$, its integral $G = \text{"}u\text{"}$ is only well-defined on the universal cover $\mathbb{C}$ of $\mathbb{C}/2\Lambda$. We'll explain all this more thoroughly later.

To conclude, I'd like to explain how

$$G = \int \frac{d\xi}{\sqrt{\xi(\xi-1)(\xi-\rho)}} \quad \text{and} \quad \int \frac{d\xi}{\sqrt{\prod_{k=1}^{4}(\xi-\xi_k)}}$$ came to be called

elliptic integrals. (They're also called Abelian integrals, after Abel.) Let $0 < \beta \le \alpha$ and consider the ellipse E defined by

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1 .$$

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† I am saying that the Riemann surface of $\sqrt{\xi(\xi-1)(\xi-\rho)}$ is $\cong$ to $\mathbb{C}/2\Lambda$. (cf. last term's Lecture 9)

<u>Claim</u>: The arc length of E is given by

$$4\alpha \int_0^1 \frac{1-k^2t^2}{\sqrt{(1-t^2)(1-k^2t^2)}}\,dt$$

(which can be considered as an integral "on" the Riemann surface of the denominator).

<u>Proof</u>: Arclength of parametrized curve $\theta \mapsto (x(\theta), y(\theta))$ from 0 to θ_0 is

$$\int_0^{\theta_0} \sqrt{(d\{x(\theta)\})^2 + (d\{y(\theta)\})^2} = \int_0^{\theta_0} \sqrt{(x'(\theta))^2 + (y'(\theta))^2}\,d\theta.$$

Here $\theta \mapsto (\alpha \sin\theta, \beta\cos\theta)$ and $\sqrt{(x')^2+(y')^2} = \sqrt{\alpha^2\cos^2\theta + \beta^2\sin^2\theta}$

$$= \alpha\sqrt{\cos^2\theta + \frac{\beta^2}{\alpha^2}\sin^2\theta} = \alpha\sqrt{(\cos^2\theta + \sin^2\theta) + \left(\frac{\beta^2}{\alpha^2}-1\right)\sin^2\theta}$$

$$= \alpha\sqrt{1 - \underbrace{\left(1-\frac{\beta^2}{\alpha^2}\right)}_{=:\,k^2}\sin^2\theta}.$$

Using symmetry to replace $\int_0^{2\pi}$ by $4\int_0^{\pi/2}$ gives $4\alpha\int_0^{\pi/2}\sqrt{1-k^2\sin^2\theta}\,d\theta$; putting $t := \sin\theta$, $dt = \cos\theta\,d\theta$, $\cos\theta = \cos(\sin^{-1}(t)) = \sqrt{1-t^2}$, changes the latter to $4\alpha\int_0^1 \sqrt{1-k^2t^2}\,\frac{dt}{\sqrt{1-t^2}} = 4\alpha\int_0^1 \frac{(1-k^2t^2)\,dt}{\sqrt{(1-k^2t^2)(1-t^2)}}$ as desired.

$\square$