Lecture 6: Harmonic functions revisited

We now want to take aim at the general Dirichlet problem, starting with the case of a disk. To that end, we need a more in-depth understanding of harmonic (and ultimately, but not today, "subharmonic") functions.

I. The mean-value property

Let U be a domain, $u: U \to \mathbb{R}$ a **continuous function.** Recall that

$$\underbrace{u \in \mathcal{H}(U)}_{\text{"u is harmonic"}} \iff u \in C^2_{\mathbb{R}}(U) \text{ and } \underset{\partial_x^2 + \partial_y^2 = 4\partial_z\partial_{\bar z}}{\boxed{\Delta}} u \equiv 0 .$$

"small-circle mean-value property"

Definition u has the SCMVP $\iff$

$$\boxed{\begin{aligned} &\forall z_0 \in U,\ \exists \varepsilon \in (0, d(z_0, U^c)) \text{ s.t.} \\ &u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})\, d\theta \quad \forall r \in (0, \varepsilon). \end{aligned}}$$

Notice that if $V \subset U$ is a subregion,

(#) u has SCMVP on $U \implies u$ has SCMVP on V.

Last term Yanli proved the mean-value theorem for harmonic functions:

$\boxed{\text{MVT}}$ $\underline{u \in \mathcal{H}(U) \implies u \text{ has SCMVP}}$.

Sketch: on a sufficiently small disk $D = D(z_0, r)$

$\exists\, f \in \mathrm{Hol}(D)$ s.t. $\mathrm{Re}(f) = u$. Then

$$f(z_0) \underset{\text{Cauchy}}{=} \frac{1}{2\pi i} \oint \frac{f(z)}{z - z_0}\, dz = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{\cancel{re^{i\theta}}} \cancel{re^{i\theta}}\, i\, d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta$$

and we can take Re of both sides. □

It turns out that the converse is also true!

$\boxed{\text{IVM}}$ $\underline{\overset{\text{(assumed continuous)}}{u} \text{ has SCMVP} \implies u \in \mathcal{H}(U)}$.

Philosophically, and in terms of how we'll use it, this is a bit like a Morera thm. for harmonic functions. For the proof, we'll need a maximum principle for functions with the SCMVP.

Lemma: Given $v \in C^0_{\mathbb{R}}(V)$ (V a region) satisfying SCMVP, and $p \in V$ s.t. $v(p) = \sup_{q \in V} v(q)$, we have $v \equiv$ Constant.

Proof: Set $\mu := \sup_{q \in V} v(q)$, $\mathcal{M} := \{ z \in V \mid v(z) = \mu \}$.

Clearly, $p \in \mathcal{M}$ ($\Rightarrow \mathcal{M} \neq \emptyset$). Given $\{z_i\} \subset \mathcal{M}$ with $\lim z_i = z_\infty \in V$, v C^0 $\Rightarrow (\mu = \lim \mu =) \lim v(z_i) = v(z_\infty)$

$\Rightarrow z_\infty \in \mathcal{M}$

$\Rightarrow \mathcal{M}$ closed in V.

So if $\mathcal{M}$ is open, then $\mathcal{M} = V$. Consider $p \in \mathcal{M}$; then $\exists\ \varepsilon \in (0, d(p, V^c))$ s.t.

$$\mu = v(p) \underset{\text{SCMVP}}{=} \frac{1}{2\pi}\int_0^{2\pi} v(p + re^{i\theta})\, d\theta \leq \frac{1}{2\pi}\int_0^{2\pi} \mu \, d\theta = \mu \quad (\forall r \in (0, \varepsilon])$$

$$\underset{v\ C^0}{\Longrightarrow} v(p + re^{i\theta}) = \mu \quad (\forall \theta, r) \Longrightarrow D(p, \varepsilon) \subset \mathcal{M}.$$

So $\mathcal{M} = V$ and $v \equiv \mu$ is constant. $\square$

Proof of "TVM": Given $\bar{D} = \bar{D}(z_0, r) \subset U$, by the solution to Dirichlet in a disk (§III below)

$$\exists\ \tilde{u} \in C^0(\bar{D}) \text{ s.t. } \begin{cases} \tilde{u}|_D \in \mathcal{H}(D) \\ \tilde{u}|_{\partial D} = u|_{\partial D}. \end{cases}$$

Consider $v := u - \tilde{u} \in C^0(\bar{D})$; clearly $v|_{\partial D} \equiv 0$, and v satisfies the SCMVP on D (using MVT $+ (*)$).

Now the Lemma $\Rightarrow$ if v attains maximum in D, then v constant (hence identically 0, since C^0 on $\bar{D}$ and zero on boundary).

So either $v \equiv 0$ or $v < 0$ in D $\Longrightarrow$ $v \leq 0$ in D.

Apply the same argument to $-v$ $\Rightarrow$ $-v \leq 0$ in D.

So $v \equiv 0$ $\Rightarrow$ $u = \tilde{u}$ $\Rightarrow$ $u|_D \in \mathcal{H}(D)$.

Since D was arbitrary, $u \in \mathcal{H}(U)$. $\square$

$\boxed{\text{Corollary}}$ Given $\{u_j\} \subset \mathcal{H}(U)$ converging uniformly on compact subsets of U to $u : U \to \mathbb{R}$, we have $u \in \mathcal{H}(U)$.

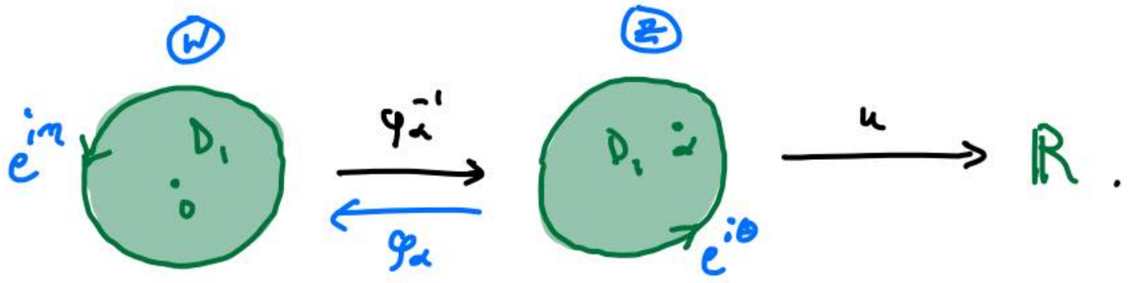
Proof: Since $\{u_j\} \subset C^0(U)$, u is C^0. Given $\bar{D} = \bar{D}(z_0, r) \subset U$, by MVT

$$\left.\begin{aligned} u_j(z_0) &= \frac{1}{2\pi}\int_0^{2\pi} u_j(z_0 + re^{i\theta})\,d\theta \\ u(z_0) &= \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})\,d\theta \end{aligned}\right\} \text{take uniform limit on } \bar{D} \text{ (both sides)}$$

$\Longrightarrow$ u has SCMVP $\Longrightarrow$ $u \in \mathcal{H}(U)$. $\square$

II. Harmonic functions on D_1

So what is $u(\alpha)$ if $\alpha \in D(z_0, r)$ isn't the center of the disk? For simplicity set $z_0 = 0$, $r = 1$,† and consider $u \in \mathcal{H}(\overline{D_1})$ (i.e. u is harmonic on an open set $U \supset \overline{D_1}$). Recall $\varphi_\alpha(z) := \frac{z-\alpha}{1-\bar{\alpha} z}$ and consider the composition



Note that

φ_α^{-1} holo. $\Longrightarrow$ $u \circ \varphi_\alpha^{-1}$ harmonic $\underset{\text{MVT}}{\Longrightarrow}$

$$u(\alpha) = (u \circ \varphi_\alpha^{-1})(0) = \frac{1}{2\pi} \int_0^{2\pi} u(\underbrace{\varphi_\alpha^{-1}(\overbrace{e^{i\eta}}^{w})}_{e^{i\theta}})\, \underbrace{d\eta}_{\substack{d\arg(w) \\ = d\arg(\varphi_\alpha(z))}}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\theta})\, d\arg \varphi_\alpha(z) .$$

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† look in Ahlfors for the general formula for an R-disk.

Since

$$d\arg\varphi_\alpha(z) \overset{\swarrow z \text{ on unit circle}}{=} -i\,d\log\varphi_\alpha(z) = -i\,\frac{\varphi_\alpha'(z)}{\varphi_\alpha(z)}\,dz$$

$$= -i\,\frac{(1-|\alpha|^2)/(1-\bar\alpha z)^{\cancel{2}}}{(z-\alpha)\,/\cancel{(1-\bar\alpha z)}}\,dz \underset{\substack{\uparrow \\ z\bar z = 1}}{=} \frac{1-|\alpha|^2}{(z-\alpha)(\bar z-\bar\alpha)}\,(-i\bar z\,dz)$$

$$= \frac{1-|\alpha|^2}{|z-\alpha|^2}\,d\theta\,,$$ we obtain

(1) $$\boxed{u(\alpha) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})\,\frac{1-|\alpha|^2}{|e^{i\theta}-\alpha|^2}\,d\theta.}$$

Noting that (for $z=e^{i\theta}$)

$$\frac{z+\alpha}{z-\alpha} = \frac{(z+\alpha)(\bar z-\bar\alpha)}{|z-\alpha|^2} = \frac{1-|\alpha|^2+i(\cdots)}{|z-\alpha|^2}\,,$$ this becomes

(2) $$\boxed{u(\alpha) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})\ \mathrm{Re}\left\{\frac{e^{i\theta}+\alpha}{e^{i\theta}-\alpha}\right\}d\theta.}$$

Finally, writing $\alpha = ae^{i\psi}$ and

$$|e^{i\theta}-ae^{i\psi}|^2 = (e^{i\theta}-ae^{i\psi})(e^{-i\theta}-ae^{-i\psi})$$

$$= 1-a\left\{e^{i(\psi-\theta)}+e^{i(\theta-\psi)}\right\}+a^2$$ yields

(3) $$\boxed{u(ae^{i\psi}) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})\,\frac{1-a^2}{1-2a\cos(\theta-\psi)+a^2}\,d\theta,}$$

where $P_a(\theta-\psi)$ is the Poisson kernel. Formulas (1)–(3) are all versions of the POISSON FORMULA.

Now, we show how to weaken the hypotheses a bit.

Theorem If $u \in C^0(\overline{D_1})$ with $u|_{D_1}$ harmonic, then

$$u(\alpha) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})\, P_a(\theta-\psi)\, d\theta$$

for all $\alpha = ae^{i\psi} \in D_1$.

Proof: Let $r \in (0,1)$, so that $u \circ \overset{\text{multiplication by } r}{\mu_r} \in \mathcal{H}(\overline{D_1})$.

Poisson $\Rightarrow$ $u(r\alpha) = (u\circ\mu_r)(\alpha) = \frac{1}{2\pi}\int_0^{2\pi} \underbrace{(u\circ\mu_r)(e^{i\theta})}_{u(re^{i\theta})} \cdot P_a(\theta-\psi)\, d\theta$

Using (uniform!) continuity of u on D_1, we get that as $r \to 1^-$, $u(re^{i\theta}) \to u(e^{i\theta})$ uniformly in $\|\cdot\|_{[0,2\pi]}$ (and thus the same for its product with the continuous function $P_a(\theta-\psi)$, which doesn't vary).

Taking limits on both sides now gives the desired result. □

Corollary Under the above hypotheses, the harmonic conjugate function of u is given (up to a constant) by

$$v(\alpha) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})\, \mathrm{Im}\left\{\frac{e^{i\theta}+\alpha}{e^{i\theta}-\alpha}\right\} d\theta .$$

Proof: Define $f(\alpha) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})\cdot\frac{e^{i\theta}+\alpha}{e^{i\theta}-\alpha}\,d\theta$

$$\left(\begin{matrix}\text{on } \partial D_1, \\ \frac{dz}{z} = d\log z \\ = d(i\theta) = i\,d\theta\end{matrix}\right) \bigcirc\!\!\!\!=\ \frac{1}{2\pi i}\oint_{\partial D_1} u(z)\cdot\frac{z+\alpha}{z-\alpha}\,\frac{dz}{z},$$

which is[†] in $\mathcal{H}ol(D_1)$ (as a function of α).
Taking real parts gives the Poisson formula (& hence u); so taking imaginary parts gives the harmonic conjugate. $\square$

III. The Dirichlet problem for D_1

There's a natural "physical" interpretation of the Poisson formula: in (1), we can see the "kernel" $\frac{1-|\alpha|^2}{|e^{i\theta}-\alpha|^2}$ (which is itself a harmonic function of α [HW]) as the "point-source-at-$e^{i\theta}$ stable temperature distribution".
More generally, if S^1 has any temperature distribution, there should extend to a stable distribution on $\overline{D}_1$.

[†] Technically, this uses the following (easy application of Morera)
<u>Lemma</u>: Let $\varphi(w,t) \in C^0(U\times[\alpha,\beta])$, with $\varphi(w,t_0) \in \mathcal{H}ol(U)$ $\forall t_0 \in [a,b]$. Then $F(w) = \int_\alpha^\beta \varphi(w,t)\,dt \in \mathcal{H}ol(U)$ (i.e. is analytic in w). [HW?]

We'll confirm this expectation mathematically now by using Poisson's formula to construct harmonic functions.

Theorem Let $f \in C^0_{\overset{\text{real-valued}}{\mathbb{R}}}(\partial D_1)$, and set

$$u(z) := \begin{cases} \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\frac{1-|z|^2}{|z-e^{i\theta}|^2}\,d\theta & , z \in D_1 \\ f(z) & , z \in \partial D_1 . \end{cases}$$

Then $u \in C^0(\overline{D_1})$ and $u|_{D_1} \in \mathcal{H}(D_1)$

The proof proceeds in two steps:

Proof Step 1 $[u|_{D_1}$ is harmonic$]$:

$u(z)$ is Re of $\frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\theta$,

which is holomorphic by the lemma in the footnote above.

Proof Step 2 [Continuity at the boundary]:

(Intuition) Poisson weighting $\to$ δ-function as z moves radially toward ∂D_1

In particular, $\frac{1-|z|^2}{|e^{i\theta}-z|^2} = 0$ for $\begin{cases} |z|=1 \\ \text{and} \\ z \neq e^{i\theta} . \end{cases}$

Actual proof: For a piecewise C^0 function $\tilde{f}$ on ∂D_1,

set $$P_{\tilde{f}}(z) := \frac{1}{2\pi}\int_0^{2\pi} \tilde{f}(e^{i\theta}) \frac{1-|z|^2}{|e^{i\theta}-z|^2}\, d\theta .$$

By the Lemma, if we write $\gamma := \overline{\{ z \in \partial D_1 \mid \tilde{f}(z) \neq 0 \}}$ then $P_{\tilde{f}} \in \mathcal{H}(\mathbb{C} \setminus \gamma)$. Fix $e^{i\theta_0} \in \partial D_1$, and assume $f(e^{i\theta_0}) = 0$. (It suffices to consider this case, as we can add a constant to u later.) Choose $\gamma_2 :=$ small open arc of ∂D_1 such that $\gamma_2 \ni e^{i\theta_0}$ and $\|f\|_{\gamma_2} < \frac{\epsilon}{2}$; set $\gamma_1 := \partial D_1 \setminus \gamma_2$. Put $\tilde{f}_i := \begin{cases} f & \text{on } \gamma_i \\ 0 & \text{on } \gamma_{2-i} \end{cases}$. Then

$$\left| P_{\tilde{f}_2}(z) \right| \leq \frac{\epsilon}{2} \cdot \frac{1}{2\pi} \int_0^{2\pi} \frac{1-|z|^2}{|z-e^{i\theta}|^2}\, d\theta \underset{(**)}{=} \frac{\epsilon}{2} \text{ for } z \in D_1 .$$

Now $\exists\, \delta > 0$ s.t. (for $z \in D_1$) i.e. $e^{i\theta} \in \gamma_1$

$$|z - e^{i\theta_0}| < \delta \implies \left\| f(e^{i\theta}) \frac{1-|z|^2}{|z-e^{i\theta}|^2} \right\|_{\gamma_1} < \epsilon/2$$

$$\implies \left| P_{\tilde{f}_1}(z) \right| < \epsilon/2 .$$

So $u = P_{\tilde{f}_1} + P_{\tilde{f}_2}$ (on D_1) $\implies$

$|u(z)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ for $z \in D(e^{i\theta_0}, \delta) \cap D_1$,

which (since $u(e^{i\theta_0}) = f(e^{i\theta_0}) = 0$) is the desired continuity statement. $\square$

IV. A note on Schwarz triangle functions

This is about a special case of an exercise in Problem Set #2, concerning the map

$$F(w) = \underbrace{e^{i\varphi}} \int_0^w \frac{d\tilde{w}}{\tilde{w}^{2/3}(w-1)^{2/3}} + \underbrace{C}$$

these allow an arbitrary rotation and translation

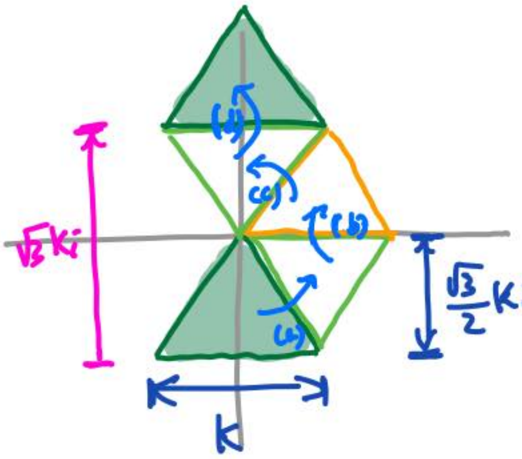
sending $\mathfrak{h}$ to an equilateral triangle in a conformal equivalence. Let

$$K := \text{side length} = \left| \int_0^1 \frac{d\tilde{w}}{\tilde{w}^{2/3}(\tilde{w}-1)^{2/3}} \right|$$

We shall determine the periods of F^{-1} by repeated Schwarz reflection. (The domain for these "reflections" alternates between $\mathfrak{h}$ and $-\mathfrak{h}$.)

- <u>first period</u>:

$$\begin{array}{ll|c}
 & & \text{function on} \\
F(w) \xrightarrow{(a)} & \bar{\zeta}_6\,\overline{\zeta_6 F(\bar{w})} = \bar{\zeta}_3\,\overline{F(\bar{w})} & -h \\
\xrightarrow{(b)} & \zeta_3 F(w) & h \\
\xrightarrow{(c)} & \zeta_6\,\overline{\bar{\zeta}_6 \zeta_3 F(\bar{w})} = \cancelto{1}{\zeta_6 \bar{\zeta}_6}\,\bar{\zeta}_3\,\overline{F(\bar{w})} & -h \\
\xrightarrow{(d)} & \overline{\overline{F(\bar{\bar{w}})} - \frac{\sqrt{3}}{2}Ki} + \frac{\sqrt{3}}{2}Ki & h \\
 & = F(w) + \underset{\text{period}}{\boxed{\sqrt{3}\,Ki}} &
\end{array}$$

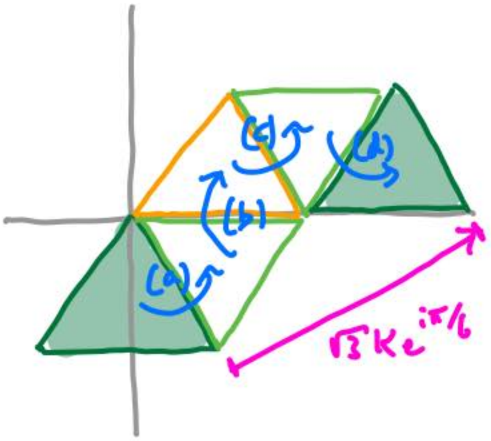


(The point is that each "reflection" is also <u>an analytic continuation of F^{-1}</u>, so the formulas prove that

$$\underbrace{F^{-1}(u+\sqrt{3}Ki) = F^{-1}(u)}.\;)$$

- <u>second period</u>:

$$\begin{array}{ll|c}
 & & \text{function on} \\
F(w) \xrightarrow{(a)+(b)} & \zeta_3 F(w) \quad \text{(as above)} & h \\
\longmapsto & \overline{\left(\left(F(\bar{w})\,\zeta_3 - K\right)\bar{\zeta}_3\right)}\,\zeta_3 + K & -h \\
 & = \zeta_3\,\overline{F(\bar{w})} - \bar{\zeta}_3 K + K & \\
\longmapsto & \overline{\left(\left(\zeta_3\,\overline{F(\bar{\bar{w}})} - \bar{\zeta}_3 K\right)\bar{\zeta}_6\right)}\,\zeta_6 + K & h \\
 & = F(w) + K(1+\zeta_6) & \\
 & = F(w) + \underset{\text{period}}{\boxed{\sqrt{3}\,Ke^{i\pi/6}}}. &
\end{array}$$



So the <u>period lattice</u> is

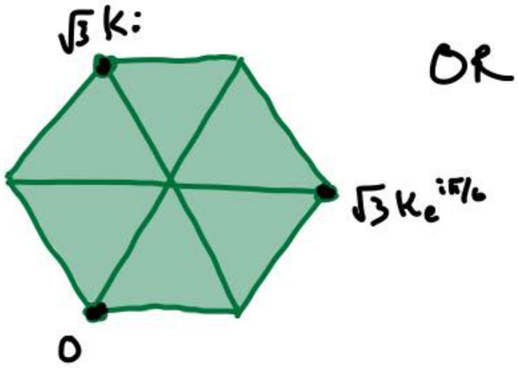
$$\Lambda = \mathbb{Z}\langle \sqrt{3}Ki,\ \sqrt{3}Ke^{i\pi/6} \rangle$$

or, after multiplying F by the constant $\frac{1}{\sqrt{3}Ke^{i\pi/6}}$,

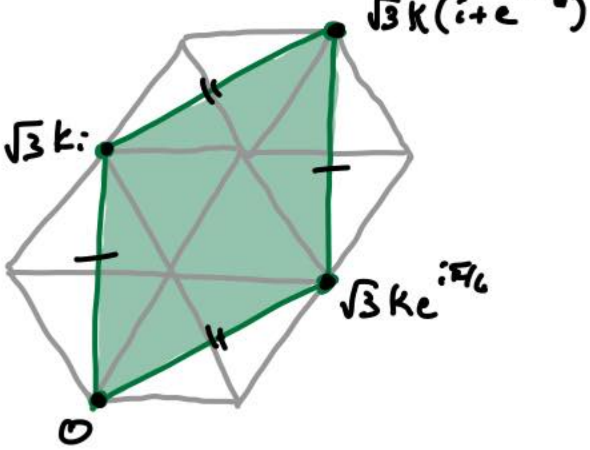
$$\Lambda \cong \mathbb{Z}\langle 1, e^{i\pi/3}\rangle.$$

$\uparrow$ from $\frac{\sqrt{3}Ki}{\sqrt{3}Ke^{i\pi/6}}$

The fundamental region for $\mathbb{C}/\Lambda$ is made up of 6 triangles and can be thought of either as



OR



Topologically $\mathbb{C}/\Lambda$ is a torus and is isomorphic to the quartic curve

$$y^3 = w^2(w-1)^2$$

(once you resolve its two singularities and compactify it). This curve has an automorphism $(y,w) \mapsto (\zeta_3 y, w)$, which corresponds to the "(a)+(b)" transformation above

$$F(w) \mapsto \zeta_3 F(w) \;;$$

that is, the automorphism of $\mathbb{C}/\Lambda$ given by

$$u \ (\mathrm{mod}\ \Lambda) \longmapsto \zeta_3 u \ (\mathrm{mod}\ \Lambda)$$

which is well-defined because $\zeta_3 \Lambda \subset \Lambda$. We say that $\mathbb{C}/\Lambda$ is a torus with *complex multiplication*; the corresponding (elliptic) algebraic curves are used extensively in cryptography.

Summing up: we have

$$F^{-1}(u+\lambda) = F^{-1}(u) \quad \forall u \in \mathbb{C},\ \lambda \in \Lambda$$

$\Longrightarrow F^{-1}$ yields a well-defined function $\mathbb{C}/\Lambda \to \mathbb{P}^1$,

and also

$$F^{-1}(\zeta_3 u) = F^{-1}(u).$$