Lecture 8: Subharmonic functions (I)

We now want to take aim at the general Dirichlet problem, solved previously for a disk. To that end, we need to introduce a more general class of functions than harmonic ones (for which nice properties still hold). After that, I'll prove and discuss the "Jensen formula" used (as an inequality) in Lecture 4.

I. What is... a subharmonic function?

The "harmonic functions" on \( \mathbb{R} \) — i.e., those killed by \( \Delta = \frac{d^2}{dx^2} \) — are just the affine functions

\[
f(x) = ax + b.
\]

On any interval, they clearly satisfy a "maximum principle": if the maximum is achieved in the interior, then the function is constant. If we are after a larger class...
of functions for which this principle holds, we might consider the convex functions: given any \( a \leq b \), these functions satisfy

\[
g(tb + (1-t)a) \leq tg(b) + (1-t)g(a) \quad \forall t \in [0,1]
\]

A "harmonic function" \( f_0 \) having the same values as \( g \) at the boundary (endpoints) of \([a,b]\).

hence for any affine \( f \) with \( f(a) \geq g(a) \), \( f(b) \geq g(b) \), we have \( g(x) \leq f(x) \) \((\forall x \in [a,b])\).

This definition generalizes easily to a complex variable setting. Notice that if a convex function \( g \) is \( C^2 \), then we can take \( \Delta g = \frac{\partial^2 g}{\partial x^2} \); if this is \(< 0\) at any point, hence on some interval \([a,b]\), we get a function \( g_0 := g - f_0 \) satisfying

\[
\begin{align*}
\Delta g_0 &< 0 \\
g_0(x) &\geq 0 \\
g_0(a) = g_0(b) = 0.
\end{align*}
\]

This is impossible (why?), so we conclude that convex functions which are \( C^2 \) satisfy \( \Delta g \geq 0 \).
Now for the complex analogue, which first appeared in work of Poincaré and Hartogs, and was then systematically studied by Riege in the 1920s. Let \( U \subset \mathbb{C} \) be open, \( f \in C^0_{\bar{\partial}}(U) \).

**Definition** \[ f \in \mathbb{H}(U) \quad (f \text{ is subharmonic on } U) \]

\[ \forall z \in U, r < \rho(z, U^c), u \in \mathbb{H}(\overline{D}(z, r)) \]

satisfying \( f \leq u \) on \( \partial D(z, r) \),

we have \( f \leq u \) on \( \overline{D}(z, r) \).

Suppose \( h \in \mathbb{H}(U) \). Is \( h \) subharmonic? Well, let \( \overline{D} = \overline{D}(z, r) \subset U \) and \( u \in \mathbb{H}(\overline{D}) \) be such that \( h \leq u \) on \( \partial \overline{D} \), i.e. \( h - u \leq 0 \) there. By the maximum principle for harmonic functions, \( h - u \leq 0 \) on \( \overline{D} \). So \( \mathbb{H}(U) \subseteq \mathbb{H}(U) \).

**Remark:** One can define superharmonic functions \( \overline{\mathbb{H}}(U) \) by reversing the inequalities in the above definition. //

The following is an analogue of the MVT for subharmonic functions, and is very useful for constructing them.
Theorem: Let $f \in C^0_\mathbb{R}(U)$, $U \subset \mathbb{C}$ open. Then

$f \in \mathcal{X}(U) \iff \frac{1}{2\pi} \int_0^{2\pi} f(p + re^{i\theta}) \, d\theta > 0 \quad \forall \overline{D}(p, r) \subset U.$

Proof: Suppose $(\star)$ holds. If $f \notin \mathcal{X}(U)$, then $\exists \overline{D}(p, r) \subset U$ and $h \in \mathcal{X}(\overline{D})$ s.t. $f \leq h$ on $\partial \overline{D}$ but $f(x_0) > h(x_0)$ for some $x_0 \in \overline{D}$. Consider $g := f - h$ on $\overline{D}$, so that $\int_{\partial \overline{D}} g = 0$. Let $M = \max_{\overline{D}}(g)$, $K = \{ z \in \overline{D} \mid g(z) = M \}$ compact.

If we take $x_0 \in \partial \overline{D}(w, \epsilon)$ with $g(x_0) < M$. Then $x_0 \in \overline{D}(w, \epsilon) \subset U$.

Since $g$ is $C^0$, there is a whole arc of $\partial \overline{D}(w, \epsilon)$ where $g < M$, so

$$\frac{1}{2\pi} \int_0^{2\pi} g(x + \epsilon e^{i\theta}) \, d\theta < M = g(w).$$

By MVT

$$\frac{1}{2\pi} \int_0^{2\pi} f(x + \epsilon e^{i\theta}) \, d\theta - h(w)$$
\[
\Rightarrow f(w) = \left(\frac{1}{\pi} \int_0^{2\pi} f(w + e^{i\theta}) \, d\theta \right) < h(w) + g(w) = f(w),
\]

a contradiction.

To do the converse, suppose \(f \in H(U)\), \(d\)

for \(D(e, \epsilon) = \overline{D} \subset U\). Let \(P: D \times \overline{D} \rightarrow \mathbb{R}\)

be the Poisson kernel \(P(z, e^{i\theta}) = \frac{1}{2\pi} \cdot \frac{\sqrt{1 - |z|^2}}{|(z - e^{i\theta})|}\) for \(D\).

Then \(h(z) = \frac{1}{2\pi} \int_0^{2\pi} P(z, e^{i\theta}) \cdot \{f(\epsilon + e^{i\theta}) + \epsilon\} \, d\theta\) defines

a continuous function on \(\overline{D}\) which is harmonic on \(D\).

Moreover, for any \(\delta \in \mathbb{D}\), \(h(\delta) = f(\delta) + \epsilon > f(\delta)\).

By continuity of \(f \& h\), \(h(\delta) > f(\delta)\) for \(\delta \in \mathbb{D}(e, \epsilon - \delta)\)

for \(\delta > 0\) (\(\delta\) depending on \(\epsilon\) sufficiently small). But then

by subharmonicity, \(f \leq h\) on \(D(e, \epsilon - \delta)\), hence

\[
f(\delta) \leq h(\delta) = \left(\frac{1}{2\pi} \int_0^{2\pi} \left\{ f(\delta + e^{i\theta}) + \epsilon \right\} \, d\theta \right)
\]

Take \(\epsilon \to 0^+\), done. \(\square\)
Corollary \[ H(U) = \bigcap H(U) \wedge \bar{H}(U) \].

Proof: \( \leq \) is obvious. By the theorem, for every small circle we have (for \( f \in \mathcal{X}(0,1) \))
\[ f(p) = \frac{1}{2\pi} \int_{0}^{2\pi} \frac{1}{1 + (p + re^{i\theta})} \, d\theta. \]
\[ \leq \text{ and } \geq \]

By “TVM”, \( f \in H(U) \).

We’ll exploit the theorem a great deal more in Monday’s lecture.
II. Counting zeroes of entire functions

In the remainder of today's lecture, we discuss (as promised) Jensen's formula and give an application. Let \( f \in \text{hol}(\mathbb{C}) \), with zeroes \( \{z_n\} \) ordered by increasing absolute value and repeated according to multiplicity (a convention we'll stick to tacitly from now on). Define \( N(r) \):

\[
N(r) := \# \text{ of zeroes with absolute value } \leq r,
\]

and assume \( f(0) \neq 0 \). We'd like to understand the asymptotic growth of \( N(r) \).

Consider \( r > 0 \) such that no \( |z_n| \) equals \( r \); the first step is to "divide out the zeroes of \( f \) inside \( B_r \)."

To do this, recall

\[
\phi(n) := \frac{n - \alpha}{1 - \alpha \bar{z}}.
\]

\[\text{e.g., if } \text{ord}_{z_n}(f) = 3 \text{ then } e_1 e_2 = e_3 = \frac{i}{2}, \text{ and if } |z_n| > 1 \text{ for } n \geq 4, \text{ then } N(1) = 3.\]
if we divide by this, then we introduce a zero at \( \frac{r}{2} \), and we don’t want to introduce zeroes in \( D_r \).

So we rescale to

\[
\varphi_{\frac{r}{2}}(\frac{2}{r}) = \frac{\varphi_{\frac{r}{2}}(\frac{x}{r})}{1 - \frac{2}{r} \varphi_{\frac{r}{2}}(\frac{x}{r})} = \frac{r(\varphi_{\frac{r}{2}}(\frac{x}{r})}{r^2 - 2x^2}
\]

and define

\[
F(x) := \frac{f(x)}{\prod_{n=1}^{N(r)} \varphi_{\frac{r}{2}}(\frac{x}{r})} \in \mathcal{K}(D_r).
\]

Clearly, \( F \) is free of zeroes in \( D_r \), and the MNP yields

\[
(*) \quad |F(0)| \leq \|F\|_{D_r}.
\]

Now

\[
F(0) = \frac{f(0)}{\prod_{n=1}^{N(r)} \varphi_{\frac{r}{2}}(0)} = \pm f(0) \prod_{n=1}^{N(r)} \frac{r}{2n}
\]

and for \( z = re^{i\theta} \)

\[
F(z) = \frac{f(z)}{\prod_{n=1}^{N(r)} \varphi_{\frac{r}{2}}(e^{i\theta})} = \frac{f(z)}{\prod_{n=1}^{N(r)} \left(\frac{e^{i\theta} - a_n}{1 - (\frac{e^{i\theta}}{2})^2}\right)} = \frac{f(z)}{\prod_{n=1}^{N(r)} \left(\frac{e^{i\theta} - e^{-i\theta}}{(e^{i\theta} - a_n)^2}\right)}
\]

\[
\implies |F(z)| = |f(z)| \text{ on } \partial D_r.
\]
So \((\zeta)\) becomes

\[
|f(0)| \leq \|f\|_{D_r} \left( \prod_{n=1}^{N(r)} \left( \frac{r}{12n} \right)^{-1} \right)
\]

or just \(D_r\)
(by MMP)

\[
\Rightarrow \log |f(0)| \leq \log \|f\|_{D_r} - \sum_{n=1}^{N(r)} \log \frac{r}{12n}
\]

\[
= \log \|f\|_{D_r} - \int_0^r \frac{N(n)}{n} \, dn
\]

\[
\Rightarrow \int_0^r \frac{N(n)}{n} \, dn \leq \log \|f\|_{D_r} - \log |f(0)|
\]

Jensen's inequality.

Example:

Let \(\lambda \geq 0\), and suppose \(f \in H^1(\Omega)\)

satisfies \(\|f\|_{D_R} \leq CR^\lambda\) for \(R \geq R_0\),

where \(C > 1\) is a constant. (For instance, \(\lambda = 0\)
corresponds to \(f\) constant; while \(\sin, \cos, \exp\) all
have \(\lambda = 1\).)

Now assume \(\lambda > 0\); then by Jensen's inequality

\[
\int_0^R \frac{N(n)}{n} \, dn \leq R^{\lambda} \log C - \log |f(0)|
\]

for \(R \geq R_0\). This is compatible with \(N(n) \leq \text{const} \times R^\lambda\)
(for \(R \gg 0\)). To actually prove this, put
Given \( f \in \text{hol}(C) \) and \( r > 0 \), define \( F \in \text{hol}(D_r) \) as above, and recall that \( F \) is zero-free on \( D_r \).

Since \( \log |F| = \text{Re} (\log F) \) is harmonic, we can apply the MVT to get...
\[
\log |F(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log |F(re^{i\theta})| \, d\theta \\
= \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| \, d\theta - \frac{1}{2\pi} \sum_{k=1}^{N_G} \int_0^{2\pi} \log \frac{e^{i\theta} - e^{-i\theta}}{1 - (e^{i\theta})^k} \, d\theta
\]

\[\iff = 0\]

\[
\Rightarrow \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| \, d\theta = \log |f(0)| + \sum_{n=1}^{N_G} \log |\frac{1}{x_n}|
\]

(Jensen’s formula).

Example // The **Mahler measure** of a polynomial \(P(x) \in \mathbb{Z}[x]\) is defined by

\[
M(P) := \exp \left\{ \frac{1}{2\pi} \int_0^{2\pi} \log |P(e^{i\theta})| \, d\theta \right\}.
\]

If \(P = \prod_{i=1}^m (x - \alpha_i)\), then by Jensen’s formula

\[
M(P) = \exp \left\{ \log |a| + \sum_{i=1}^m \log |\alpha_i| - \sum_{i=1}^m \log |\frac{1}{\alpha_i}| \right\}
\]

\[= |a| \prod_{i=1}^m \max \{1, |\alpha_i|\}.\]

For (products of) cyclotomic polynomials, \(M(P)\) is clearly 1.

A major unsolved problem in number theory is the
Conjecture (Lehmer): For non-cyclotomic $P \in \mathbb{Z}[x]$, 
\[ M(P) \geq M\left(1 - x + x^3 - x^4 + x^5 - x^6 + x^7 - x^9 + x^{10}\right) \approx 1.1762. \]
\[ = \cdot P_0(x). \]
(Note that all but 2 of $P_0$'s roots lie on $\mathbb{D}_1$, so only one root contributes to the value of $M(P_0)$.)

We actually cheated a bit in the last example, because we have not yet addressed what happens if some roots of $f$ lie on $\mathbb{D}_r$. First, this is not a problem for convergence of the LHS of Jensen's formula, because
\[ \left| \int_{\mathcal{E}} \log |x| \, dx \right| = \left| \left( \int \log |x| \right)_{\mathcal{E}} \right| = \left| e \log e - e \right| < \infty. \]
(i.e. \( \lim_{\epsilon \to 0} (\log e - e) \), using \( \lim_{\epsilon \to 0} e \log \epsilon \epsilon = 0 \))

Further, the RHS of LHS are both continuous in $r$:

RHS: $\log |\frac{e}{2\pi}| = 0$ when $|2\pi| = r$, so that when the # of terms in the sum jumps with $N(r)$, the value of the sum does not.

LHS: need to show that the difference of $S$'s of $\log |f|$ over small arcs (as shown) limits to 0, for which it suffices to check the same thing for a small circle about the pole, which is just $\lim_{\epsilon \to 0} \int_{\mathbb{D}(e, \epsilon)} \log |f| / 2\pi = 2\pi e \log e \to 0$.