Lecture 8: Subharmonic functions (I)

We now want to take aim at the general Dirichlet problem, solved previously for a disk. To that end, we need to introduce a more general class of functions than harmonic ones (for which nice properties still hold). After that, I'll prove & discuss the "Jensen formula" used (as an inequality) in Lecture 4.

I. What is... a subharmonic function?

The "harmonic functions" on $\mathbb{R}$ — i.e., those killed by $\Delta = \partial_x^2$ — are just the affine functions

$$f(x) = ax + b.$$

On any interval, they clearly satisfy a "maximum principle": if the maximum is achieved in the interior, then the function is constant. If we are after a larger class of functions for which this principle holds, we might consider the convex functions: given any $a \le b$, these functions satisfy

$$g(tb + (1-t)a) \le \underbrace{tg(b) + (1-t)g(a)}_{\substack{\text{"harmonic function" } f_0 \\ \text{having the same values as } g \\ \text{at the boundary (endpoints) of } [a,b].}} \quad \forall t \in [0,1]$$

hence for any affine f with $f(a) \ge g(a)$, $f(b) \ge g(b)$, we have $g(x) \le f(x)$ $(\forall x \in [a,b])$.

This definition generalizes easily to a complex variable setting. Notice that if a convex function g is C^2, then we can take $\Delta g = \frac{\partial^2 g}{\partial x^2}$; if this is < 0 at any point, hence on some interval (a,b), we get a function $g_0 := g - f_0$ satisfying

$$\left.\begin{array}{l} \Delta g_0 < 0 \\ g_0 \le 0 \end{array}\right\} \text{ on } [a,b], \quad g_0(a) = g_0(b) = 0.$$

This is impossible (why?), so we conclude that convex functions which are C^2 satisfy $\Delta g \ge 0$.

Now for the complex analogue, which first appeared in work of Poincaré and Hartogs, and was then systematically studied by Riesz in the 1920s. Let $U \subset \mathbb{C}$ be open, $f \in C^0_{\mathbb{R}}(U)$.

Definition $f \in \underline{\mathcal{H}}(U)$ (f is subharmonic on U)

$$\iff \boxed{\begin{array}{l}\forall z_0 \in U,\ r \in (0, d(z_0, U^c)),\ u \in \mathcal{H}(\overline{D}(z_0,r)) \text{ satisfying } f \leq u \text{ on } \partial\overline{D}(z_0,r), \\ \text{we have } f \leq u \text{ on } D(z_0,r).\end{array}}$$

Suppose $h \in \mathcal{H}(U)$. Is h subharmonic?? Well, let $\overline{D} = \overline{D}(z_0, r) \subset U$ and $u \in \mathcal{H}(\overline{D})$ be such that $h \leq u$ on $\partial \overline{D}$, i.e. $h - u \leq 0$ there. By the maximum principle for harmonic functions, $h - u \leq 0$ on $\overline{D}$. So

$$\mathcal{H}(U) \subseteq \underline{\mathcal{H}}(U).$$

Remark// One can define superharmonic functions $\overline{\mathcal{H}}(U)$ by reversing the inequalities in the above definition. //

The following is an analogue of the MVT for subharmonic functions, and is very useful for constructing them.

Theorem Let $f \in C^0_{\mathbb{R}}(U)$, $U \subset \mathbb{C}$ open. Then

$$f \in \underline{\mathcal{H}}(U) \iff \underbrace{\boxed{f(p) \leq \frac{1}{2\pi}\int_0^{2\pi} f(p + re^{i\theta})\,d\theta} \ \forall\, \overline{D}(p,r) \subset U}_{(*)}.$$

Proof: Suppose $(*)$ holds. If $f \notin \underline{\mathcal{H}}(U)$, then $\exists$ $\overline{D}(q,s) =: \overline{D}' \subset U$ and $h \in \mathcal{H}(\overline{D}')$ s.t. $f \leq h$ on $\partial D'$ but $f(z_0) > h(z_0)$ for some $z_0 \in D'$. Consider $g := f - h$ on $\overline{D}'$, so that $\begin{cases} g \leq 0 & \text{on } \partial D' \\ g(z_0) > 0 \end{cases}$.

Let $M = \max_{\overline{D}'}(g)$, $K = \{z \in \overline{D}' \mid g(z) = M\} \underset{\text{compact}}{\subset} D'$.

If $w \in \partial K$ then $\exists$ $z_0 \in \partial D(w, \epsilon)$ with $g(z_0) < M$.

(here $\overline{D}(w,\epsilon) \subset U$)

Since g is C^0, there is a whole arc of $\partial D(w,\epsilon)$ where $g < M$, so

$$\frac{1}{2\pi}\int_0^{2\pi} g(w + \epsilon e^{i\theta})\,d\theta < M = g(w)$$

$\| \text{ MVT}$

$$\frac{1}{2\pi}\int_0^{2\pi} f(w + \epsilon e^{i\theta})\,d\theta - h(w)$$

$$\Rightarrow \quad f(w) \overset{(*)}{\leq} \frac{1}{2\pi}\int_0^{2\pi} f(w+\epsilon e^{i\theta})\,d\theta \;<\; h(w)+g(w) = f(w),$$

a contradiction.

To do the converse, suppose $f \in \underline{\mathcal{H}}(U)$, & fix $\overline{D}(q,s) =: \overline{D} \subset U$. Let $P: D \times \partial D \to \mathbb{R}$ be the Poisson kernel $P(z, e^{i\theta}) = \frac{1}{2\pi} \cdot \frac{s^2 - |z-q|^2}{|(z-q) - se^{i\theta}|^2}$ for D. Then $h(z) := \frac{1}{2\pi}\int_0^{2\pi} P(z, e^{i\theta}) \cdot \{f(q+se^{i\theta}) + \epsilon\}\,d\theta$ defines a continuous function on $\overline{D}$ which is harmonic on D. Moreover, for any $\xi \in \partial D$, $h(\xi) = f(\xi) + \epsilon > f(\xi)$. By continuity of f & h, $h(\xi) > f(\xi)$ for $\xi \in \partial D(q, s-\delta)$ for $\delta > 0$ (δ depending on ϵ) suff. small. But then by subharmonicity, $f \leq h$ on $D(q, s-\delta)$, hence

$$f(q) \leq h(q) = \underset{P(q,e^{i\theta})}{\underset{\shortparallel}{\frac{1}{2\pi}}}\int_0^{2\pi} \{f(q+se^{i\theta}) + \epsilon\}\,d\theta \quad \forall \epsilon > 0.$$

Take $\epsilon \to 0^+$, done. $\square$

$\boxed{\text{Corollary}}$ $\mathcal{H}(U) = \underline{\mathcal{H}}(U) \cap \overline{\mathcal{H}}(U)$.

Proof: "$\subseteq$" is obvious. By the Theorem, for every small circle we have (for $f \in \underline{\mathcal{H}} \cap \overline{\mathcal{H}}$)

$$f(p) \underset{\le \text{ and } \ge}{=} \frac{1}{2\pi}\int_0^{2\pi} f(p+re^{i\theta})\,d\theta.$$

By "TVM", $f \in \mathcal{H}(U)$. $\square$

We'll exploit the Theorem a great deal more in Monday's lecture.

II. Counting zeroes of entire functions

In the remainder of today's lecture, we discuss (as promised) Jensen's formula and give an application.

Let $f \in \mathcal{H}ol(\mathbb{C})$, with zeroes $\{z_n\}$ ordered by increasing absolute value and <u>repeated according to multiplicity</u> (a convention we'll stick to tacitly from now on). Define[†]

(counted w/ multiplicity!)

$$N(r) := \#\text{ of zeroes with absolute value} \leq r,$$

and assume $f(0) \neq 0$. <u>We'd like to understand the asymptotic growth of $N(r)$.</u>

Consider $r > 0$ such that no $|a_n|$ equals r; the first step is to "divide out the zeroes of f inside D_r". To do this, recall

$$\varphi_\alpha(z) := \frac{z-\alpha}{1-\bar{\alpha}z} \quad ;$$

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† e.g., if $\mathrm{ord}_{i/2}(f) = 3$ then $z_1 = z_2 = z_3 = \frac{i}{2}$, and if $|z_n| > 1$ for $n \geq 4$, then $N(1) = 3$.

if we divide by this, then we introduce a zero at $1/\bar{\alpha}$, and we don't want to introduce zeroes in D_r.

So we rescale to

since $\left|r^2/\bar{\alpha}\right| > r^2/r = r$

$$\varphi_{\alpha/r}(z/r) = \frac{z/r - \alpha/r}{1 - \bar{\alpha} z/r^2} = \frac{r(z-\alpha)}{r^2 - \bar{\alpha} z} \in Hol(\overline{D_r})$$

$\alpha \in D_r$

and define

$$F(z) := \frac{f(z)}{\prod_{n=1}^{N(r)} \varphi_{\frac{z_n}{r}}\left(\frac{z}{r}\right)} \in Hol(\overline{D_r}).$$

Clearly, F is free of zeroes in $\overline{D_r}$, and the MMP yields

$$(*) \qquad |F(0)| \leq \|F\|_{\partial D_r}.$$

Now

$$F(0) = \frac{f(0)}{\prod_{n=1}^{N(r)} \varphi_{\frac{z_n}{r}}(0)} = \pm f(0) \cdot \prod_{n=1}^{N(r)} \frac{r}{z_n},$$

and for $z = re^{i\theta}$

$$F(z) = \frac{f(z)}{\prod_{n=1}^{N(r)} \varphi_{z_n/r}(e^{i\theta})} = \frac{f(z)}{\prod_{n=1}^{N(r)} \left(\frac{e^{i\theta} - z_n/r}{1 - (\bar{z}_n/r)e^{i\theta}}\right)} = \frac{f(z)}{\prod_{n=1}^{N(r)} e^{-i\theta} \frac{e^{i\theta} - \frac{z_n}{r}}{\left(\overline{e^{i\theta} - \frac{z_n}{r}}\right)}}$$

$$\implies |F(z)| = |f(z)| \text{ on } \partial D_r.$$

So (*) becomes

$$|f(0)| \le \|f\|_{\partial D_r} \left(\prod_{n=1}^{N(r)} \frac{r}{|z_n|} \right)^{-1}$$

or just $\overline{D_r}$ (by MMP)

$$\Longrightarrow \log|f(0)| \le \log\|f\|_{D_r} - \sum_{n=1}^{N(r)} \log\frac{r}{|z_n|} \quad = \int_{|z_n|}^{r} \frac{dx}{x}$$

$$= \log\|f\|_{D_r} - \int_0^r \frac{N(x)}{x}\,dx$$

$$\Longrightarrow \boxed{\int_0^r \frac{N(x)}{x}\,dx \le \log\|f\|_{D_r} - \log|f(0)|}$$ Jensen's inequality.

Example // Let $\lambda \ge 0$, and suppose $f \in \mathcal{H}ol(\mathbb{C})$ satisfies $\|f\|_{D_R} \le C^{R^\lambda}$ for $R \ge R_0$, where $C > 1$ is a constant. (For instance, $\lambda = 0$ corresponds to f constant; while sin, cos, exp all have $\lambda = 1$.)

Now assume $\lambda > 0$: then by Jensen's inequality

$$\int_0^R \frac{N(x)}{x}\,dx \le R^\lambda \log C - \log|f(0)|$$

for $R \ge R_0$. This is compatible with $N(x) \le \text{const.} \times R^\lambda$ (for $R \gg 0$). To actually prove this, put

$$g_R := \frac{f}{\prod_{n=1}^{N(R)} (z - z_n)} \cdot \prod_{n=1}^{N(R)} z_n \quad \left(\in \mathcal{Hol}(\mathbb{C}) \right)$$

and note $g_R(0) = \pm f(0)$. Then the MMP $\Longrightarrow$

$$|f(0)| = |g_R(0)| \leq \|g_R\|_{\partial D_{(1+e)R}} \leq \frac{\|f\|_{D_{(1+e)R}}}{(eR)^{N(R)}} \cdot R^{N(R)} = \frac{\|f\|_{D_{(1+e)R}}}{e^{N(R)}}$$

$$\Longrightarrow \quad e^{N(R)} \leq \frac{\|f\|_{D_{(1+e)R}}}{|f(0)|} \leq \frac{C^{((1+e)R)^{\lambda}}}{|f(0)|}$$

take log

$$\Longrightarrow \quad N(R) \leq \{(1+e)^{\lambda} \log C\} \times R^{\lambda} - \log |f(0)|$$

$$\leq \text{Const} \times R^{\lambda} \quad \text{for } R \gg 0.$$

We say $N(R) = O(R^{\lambda})$. (Note that this is indeed true for sin, cos, exp and $\lambda = 1$!) //

III. Jensen's formula and Mahler measure

Given $f \in \mathcal{Hol}(\mathbb{C})$ and $r > 0$, define $F \in \mathcal{Hol}(\bar{D}_r)$ as above, and recall that F is zero-free on D_r.

Since

$$\log|F| = \mathrm{Re}(\log F)$$

is harmonic, we can apply the MVT to get

$$\log|F(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|F(re^{i\theta})|\,d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\log|f(re^{i\theta})|\,d\theta - \underbrace{\frac{1}{2\pi}\sum_{n=1}^{N(r)}\int_0^{2\pi}\log\underbrace{\left|\frac{e^{i\theta}-z_n/r}{1-(\bar{z}_n/r)e^{i\theta}}\right|}_{=1}d\theta}_{=0}$$

$$\Longrightarrow \boxed{\frac{1}{2\pi}\int_0^{2\pi}\log|f(re^{i\theta})|\,d\theta = \log|f(0)| + \sum_{n=1}^{N(r)}\log\left|\frac{r}{z_n}\right|}$$

(Jensen's formula) .

Example // The Mahler measure of a polynomial $P(x)\in\mathbb{Z}[x]$ is defined by

$$M(P) := \exp\left\{\frac{1}{2\pi}\int_0^{2\pi}\log|P(e^{i\theta})|\,d\theta\right\}.$$

If $P = a\prod_{i=1}^{m}(x-\alpha_i)$, then by Jensen's formula

$$M(P) = \exp\left\{\log|a| + \sum_{i=1}^{n}\log|a_i| - \sum_{i=1}^{n}\log^{-}|a_i|\right\}$$

$$= |a|\prod_{i=1}^{m}\max\{1,|a_i|\}.$$

takes value 0 if $a_i \notin D_1$, $\log(a_i)$ if $a_i\in D_1$

For (products of) cyclotomic polynomials, $M(P)$ is clearly 1.

A major unsolved problem in number theory is the

Conjecture (Lehmer): For noncyclotomic $P \in \mathbb{Z}[x]$,

$$M(P) \geq M(\underbrace{1-x+x^3-x^4+x^5-x^6+x^7-x^9+x^{10}}_{=:\, P_0(x)}) \approx 1.1762.$$

(Note that all but 2 of P_0's roots lie on ∂D_1, so only one root contributes to the value of $M(P_0)$!) //

We actually cheated a bit in the last example, because we have not yet addressed what happens if some roots of f lie on ∂D_r. First, this is not a problem for convergence of the LHS of Jensen's formula, because

$$\left|\int_0^\epsilon \log x\,dx\right| = \left|\underbrace{\text{``}(x\log x - x)\Big|_0^\epsilon\text{''}}\right| = \left|\epsilon\log\epsilon - \epsilon\right| < \infty.$$

(i.e. $\lim\limits_{\epsilon_0\to 0}(x\log x - x)\Big|_{\epsilon_0}^{\epsilon}$, using $\lim\limits_{\epsilon_0\to 0}\epsilon_0\log\epsilon_0 = 0$)

Further, the RHS & LHS are both continuous in r:

RHS: $\log\left|\frac{r}{z_n}\right| = 0$ when $|z_n| = r$, so that when the # of terms in the sum jumps with $N(r)$, the value of the sum does not.

LHS: need to show that the difference of $\int$'s of $\log|f|$ over small arcs (as shown) limits to 0, for which it suffices to check the same thing for a small circle about the zero, which is just $\lim\limits_{\epsilon\to 0}\int_{\partial D(z_0,\epsilon)} \log\epsilon \; d|z| = 2\pi\epsilon\log\epsilon \to 0$.