Lecture 9: Subharmonic functions II

Recall that \( f \in C^0_{\text{loc}}(U) \) is \textbf{subharmonic} \( (f \in \mathcal{K}(U)) \) for all \( \overline{D} \subset U \) and \( u \in \mathcal{H}(\overline{D}) \) s.t. \( (f-u)_D \leq 0 \), we have \( (f-u)_D \leq 0 \).

\[ f(p) \leq \frac{1}{2\pi} \int_0^{2\pi} f(p+re^{i\theta}) \, d\theta \quad \forall \overline{D}(p,r) \subset U. \]

(Here, as usual, \( \mathcal{H}(\overline{D}) \) denotes harmonic functions on the closed disk \( \overline{D} \), which is to say, functions on \( \overline{D} \) which are restrictions to \( \overline{D} \) of harmonic functions on any open set containing \( \overline{D} \).) In this lecture we'll deal with some examples and applications of these functions.

I. Examples

1. \textbf{Linear combinations:} If \( f_1, f_2 \in \mathcal{K}(U) \), and \( c_1, c_2 \in \mathbb{R}_{\geq 0} \), then \( c_1 f_1 + c_2 f_2 \in \mathcal{K}(U) \).

   \textbf{Proof:} Immediate consequence of SMVT.
In the last lecture we saw that harmonic functions are trivially subharmonic. Now, harmonic functions are smooth \( C^{\infty} \) — what if we ask for all \( C^2 \) subharmonic functions?

2. \( f \in C^2_{\mathbb{R}}(U) \) is subharmonic \( \iff \Delta f \geq 0 \).

**Proof:** \( (\Rightarrow) \): Let \( p \in U \). We have,

for some \( G \in C^2_{\mathbb{R}}(U) \) with \( \lim_{r \to 0^+} \frac{G(p+re^{i\theta})}{r^2} = 0 \),

\[
f(p) \leq \frac{1}{2\pi} \int_0^{2\pi} f(p + re^{i\theta}) \, d\theta
\]

\( \text{Smvtr} \)

\[
= \frac{1}{2\pi} \int_0^{2\pi} \left\{ f(p) + f_x(p) r \cos \theta + f_y(p) r \sin \theta \\
+ \frac{f_{xx}(p)}{2} r^2 \cos^2 \theta + \frac{f_{yy}(p)}{2} r^2 \sin^2 \theta \\
+ \frac{f_{xy}(p)}{r} r^2 \cos \theta \sin \theta + G(p+re^{i\theta}) \right\} \, d\theta
\]

\[
= f(p) + \frac{r^2}{4} \left( f_{xx}(p) + f_{yy}(p) \right) + \frac{r^2}{2\pi} \int_0^{2\pi} \frac{G(p+re^{i\theta})}{r^2} \, d\theta
\]

\[
= f(p) + \frac{r^2}{4} \left( (\Delta f)(p) + \frac{2}{\pi} \int_0^{2\pi} \frac{G(p+re^{i\theta})}{r^2} \, d\theta \right)
\]

Taking \( r \) sufficiently small

that the 2nd term in brackets has smaller absolute
value than the first, we see that \( (\Delta f)(p) \geq 0 \).

\[
\Leftarrow: \quad \text{If } f \text{ has a local maximum at } p, \text{ then } f_{xx}(p), f_{yy}(p) \leq 0. \text{ So } (\Delta f)(p) > 0 \Rightarrow f \text{ cannot have a local maximum at } p.
\]

So assume \( \Delta f > 0 \) on \( U \). If \( h \in C^{1}(\overline{B}(p,r)) \) with \( f \leq h \) on \( \partial D(p,r) \), then \( \Delta (f-h) = \Delta f > 0 \Rightarrow f-h \) cannot have a local maximum anywhere on \( D \).

Were we to have \( f \leq h \) somewhere in \( D(p,r) \), then \( f-h \) attains a (positive) maximum at some point \( q \in D(p,r) \), a contradiction. Conclude that \( f \leq h \) on \( D \). So \( f \in C^{1}(U) \) if \( \Delta f > 0 \).

Now assume only \( \Delta f \geq 0 \). Given \( \varepsilon > 0 \),

\[
\Delta (f + \varepsilon |x|^2) > 0 \text{ everywhere on } U
\]

(where we used \( \Delta |x|^2 = \Delta (x^2 + y^2) = 4 > 0 \)). So \( f + \varepsilon |x|^2 \in C^{1}(U) \). Given \( h \in C^{1}(\overline{D}) \) with \( h \geq f \) on \( \partial D \) \( (i = \partial D(p,r)) \), let \( \tilde{h}_\varepsilon := h + \varepsilon r^2 \) \( (r \in C^{1}(\overline{D})) \).
so that \( \tilde{h}_e \geq f + \epsilon |x|^2 \) on \( \partial D \). By (8) and (9),
\[ f + \epsilon |x|^2 \in X(U), \]
and so \( \tilde{h}_e \geq f + \epsilon |x|^2 \) on \( D \) for all \( \epsilon > 0 \). Taking \( \epsilon \to 0^+ \), we find \( h \geq f \) on \( D \).

Example 0
\[
f(x, y) = \overbrace{x^2 + y^2}^{\geq 0} + C \text{ is subharmonic.}
\]

Example 1
Say \( h \in X(U) \). Then
\[
\Delta h^2 = \partial_x^2 h^2 + \partial_y^2 h^2 = 2x \partial_x h + \partial_y 2h_y
\]
\[
= 2h_x^2 + 2h_y^2 + 2 h \Delta h \geq 0
\]
\[
\implies h^2 \in X(U).
\]

Example 2
Say \( f \in X(U) \), \( f \geq 0 \), \( f \in C^2(U) \).
Let's try the same approach:
\[
\Delta f^k = \partial_x^2 f^k + \partial_y^2 f^k = \partial_x h f^{k-1} f_x + \partial_y h f^{k-1} f_y
\]
\[
= h (k-1) f^{k-2} \{ f_x^2 + f_y^2 \} + h f^{k-1} \Delta f \geq 0
\]

\(\uparrow\) writing \( \partial_x = \frac{\partial}{\partial x} \), \( \partial_y = \frac{\partial}{\partial y} \)
\[ f^{2k} \in X(U). \]
So \( |z|^{2k} \) is subharmonic.

\[ \boxed{3} \]
(a) \( f \in X(U), \ g : \mathbb{R} \to \mathbb{R} \) nondecreasing & convex on \( f(U) \)
\[ \Rightarrow g \circ f \in X(U). \]
(b) \( f \in X(U), \ g : \mathbb{R} \to \mathbb{R} \) convex on \( f(U) \)
\[ \Rightarrow g \circ f \in X(U). \]

These are certainly suggested by the the examples above — e.g. \( g(x) = x^2 \) is convex, and increasing on \( \mathbb{R}_{\geq 0} \).

To see how to approach the situation where \( g \circ f \notin C^2 \), consider

\[ \boxed{\text{Example 3}} \]
Let \( h \in X(U) \) be given: then
\[ |h(p)| = \left| \frac{1}{2\pi} \int_0^{2\pi} h(p + re^{i\theta}) \, d\theta \right| \leq \frac{1}{2\pi} \int_0^{2\pi} |h(p + re^{i\theta})| \, d\theta \]
\[ \Rightarrow |h| \in X(U). \]

The generalization of the above inequality is given by the
Lemma: \( f \in C^0([a,b]) \), \( q : \mathbb{R} \to \mathbb{R} \), convex (at least on image \((f)\)) \( \Rightarrow \)

\[
\left( \frac{1}{b-a} \int_a^b f(x) \, dx \right) q \leq \frac{1}{b-a} \int_a^b (q \circ f)(x) \, dx.
\]

\((\#)\)

Proof: If \( q(x) = Ax + B \), then

\[
\text{LHS } (\#) = \frac{A}{b-a} \int_a^b f(x) \, dx + B \quad \text{and}
\]

\[
\text{RHS } (\#) = \frac{1}{b-a} \int_a^b (Af(x) + B) \, dx
\]

are clearly equal \((\dagger)\).

Now let \( q \) be more general, and \((x_0, q(x_0))\) a point on its graph; set \( L_{x_0}(x) = \mu(x - x_0) + q(x_0). \) If we do not have \( q \geq L_{x_0} \) for any \( \mu \), then there exists \( \mu \) for which there are \( x < x_0 \) and \( x_2 > x_0 \) with \( q(x_i) < L_{x_0}(x_i) \) \((i = 1, 2)\), which contradicts the definition of convexity. So \( \exists \) such a \( \mu \); set \( L := L_{x_0}. \)

Specialize to \( x_0 := \frac{1}{b-a} \int_a^b f(x) \, dx \). Then

\[
q(x_0) = L(x_0) = \frac{1}{b-a} \int_a^b L(f(x)) \, dx \leq \frac{1}{b-a} \int_a^b q(f(x)) \, dx.
\]

\((\dagger)\)
Proof of (3a): Use SMVT:

\[ \varphi(f(p)) \leq \varphi\left(\frac{1}{2\pi} \int_0^{2\pi} f(p+re^{i\theta}) \, d\theta\right) \leq \frac{1}{2\pi} \int_0^{2\pi} \varphi(f(p+re^{i\theta})) \, d\theta. \]

\( f \) nondecreasing, \( f \in \mathcal{X}(U) \) \hspace{1cm} \boxed{\text{Lemma}}

Proof of (3b):

\[ \varphi(f(p)) = \varphi\left(\frac{1}{2\pi} \int_0^{2\pi} f(p+re^{i\theta}) \, d\theta\right) \leq \frac{1}{2\pi} \int_0^{2\pi} \varphi(f(p+re^{i\theta})) \, d\theta. \]  

\( \text{MVT} \) \hspace{1cm} \boxed{\text{Lemma}}

Example 4: \( f \in \mathcal{X}(U) \Rightarrow e^f \in \mathcal{X}(U) \).

If \( F \neq 0 \) on \( U \) is holomorphic, then

\[ \log|F| \in \mathcal{X}(U) \Rightarrow |F| \in \mathcal{X}(U). \]

\boxed{\text{}}
II. Maximum principle

Theorem (MP) Let $U \subset \mathbb{C}$ be a region (= conn. open set), $f \in \mathcal{H}(U)$ a subharmonic function, and $p \in U$ s.t. $f(p) \geq f(z)$ $(\forall z \in U)$. Then $f$ is constant.

Proof: Let $K := \{z \in U \mid f(z) = f(p)\}; \ p \in K \Rightarrow K \neq \emptyset$.

- $K$ is open: $f(p) \leq \frac{1}{2\pi} \int_{C} f(p+re^{i\theta}) d\theta \leq \frac{1}{2\pi} \int_{C} f(p) d\theta = f(p)$

  $d = f \circ C \Rightarrow f(p+re^{i\theta}) = f(p) \quad \forall r, \theta$.

- $K$ is closed: given $\{z_{n}\} \subset K$ sequence w/limit $z \in U$.

  $f \circ C \Rightarrow f(p) = f(z_{n}) \rightarrow f(z)$

  $\Rightarrow f(z) = f(p) \Rightarrow z \in K$.

Therefore $K = U$.

Corollary 1 Given: $f \in \mathcal{H}(U)$; $U_{0}$ bounded region with $U_{0} \subset U$; $h \in \mathcal{H}(U_{0})$ with $f \leq h$ on $\partial U_{0}$.

Then $f \leq h$ on $U_{0}$.

Remark: This goes beyond the definition $U_{0}$ is more general than a disk.
Proof: \( f - h \in H(U_0) \). If \( f \) isn't \( \leq h \) on \( U_0 \), then \( f - h \) has a maximum on \( U_0 \) \[ \Rightarrow f - h \equiv c > 0 \]. This contradicts that \( f - h \leq 0 \) on \( \partial U \) and \( f - h \in C \).

[Corollary 2] Subharmonicity is a local property.

Sketch: Suppose \( f \notin \mathcal{H}(U) \) but \( f \) is subharmonic in every neighborhood of a covering \( \{ N_a \} \). Then \( \exists \bar{D} \subset U \) and \( h \in \mathcal{H}(\bar{D}) \) such that \( f - h \leq 0 \) on \( \partial \bar{D} \) and \( f - h \neq 0 \) on \( \bar{D} \)

\[ \Rightarrow f - h \) has maximum at some \( p \in D \)
\[ \Rightarrow f - h \) has maximum in nbhd. \( N \) of \( p \)

\[ \Rightarrow f - h \mid_{N_p} = C \]
\[ \Rightarrow f - h \mid_{D} = C (> 0) \]. Contradiction.

Example 5: \( f_1, f_2 \in \mathcal{H}(U) \) \( \Rightarrow f = \max \{ f_1, f_2 \} \in \mathcal{H}(U) \).

Proof: Let \( \bar{D} \subset U \), \( u \in \mathcal{H}(\bar{D}) \), \( u \geq f \) on \( \partial D \).

Then \( u \geq f_1, f_2 \) on \( \partial D \) \[ \Rightarrow u \geq \max \{ f_1, f_2 \} \) on \( D \).
Example 6: \( f \in X(U), \overline{D} \subseteq U, \quad f(x) := \begin{cases} p_f(x), & x \in D \\ f(x), & x \notin D \end{cases} \)

\( \Rightarrow \bar{f} \in X(U). \) [Here \( p_f \in X(D) \cap C^0(\overline{D}) \) is given by integrating \( f \) against the Poisson kernel.]

Proof: Clearly \( \bar{f} \) is continuous as \( \bar{f} = p_f \) on \( \overline{D} \) (cf. Lecture 6). \( p_f \) isn't defined outside \( \overline{D} \) so we can't just "use Example 5." But since \( \bar{f} \) is harmonic on \( D \) and subharmonic on \( U \setminus D \), we only need to check (by Case 2) small disks in \( \overline{U} = U \) with \( \partial U \cap D \neq \emptyset \).

Suppose (for a contradiction) that \( \exists \bar{h} \in X(\overline{U}) \) s.t. \( \bar{f} - \bar{h} \leq 0 \) on \( \partial \overline{U} \) but \( \bar{f} - \bar{h} > 0 \) somewhere in \( \overline{U} \).

Now \( p_f \geq \bar{f} \) on \( D \) simply because \( p_f \) is harmonic on \( D \) (\& \( C^0 \) on \( \overline{D} \)) and \( \bar{f} \) is subharmonic (cf. Lecture 8); it follows that \( \bar{f} \geq \bar{f} \) everywhere and so \( \bar{f} - \bar{h} \leq 0 \) on \( \partial \overline{U} \). Again invoking subharmoncity of \( f \), we cannot have \( \bar{f} - \bar{h} > 0 \) anywhere in \( N \cap (U \setminus D) \), as \( \bar{f} = \bar{f} \) there. So \( \bar{f} - \bar{h} \leq 0 \) on \( \partial (N \cap D) \), and >0 somewhere in \( N \cap D \). But \( \bar{f} - \bar{h} \) is harmonic on \( N \cap D \), and continuous on \( N \cap D \), so in view of the maximum principle this is impossible. \( \square \)
Example 7: \[ |z| \text{ is subharmonic } \Rightarrow |z|^k \text{ subharmonic (when)} \]

Indeed, we know this away from 0, and by Cor. 2 one only needs to know that \( |z| - h \) (h harmonic) has a max. at 0. Of course, one could then have

\[ -h(0) \geq \frac{1}{2\pi} \int_0^{2\pi} (|re^{i\theta}| - h(re^{i\theta})) \, d\theta = r - h(0), \]

which is advantageous.

Remark: Most definitions of subharmonic functions relax the condition that \( f \) be in \( C^0(U) \), replacing continuity by upper semicontinuity and allowing values in \( \mathbb{R} \setminus \{-\infty\} \). So for example, \( \log |z| \) is subharmonic (in this more general sense) on \( U \), because it is actually continuous as a function into \( \mathbb{R} \setminus \{-\infty\} \), harmonic on \( C \setminus \{0\} \), and satisfies the defining property of subharmonic functions (or quasiregularity, the "sub-mean-value" property) on a neighborhood of 0.