# Lecture 9: Subharmonic functions II

Recall that $f \in C^0_{\mathbb{R}}(U)$ is subharmonic $(f \in \underline{\mathcal{H}}(U))$

$\overset{\phantom{x}}{\underset{\text{def.}}{\iff}}$ for all $\bar{D} \subset U$ and $u \in \mathcal{H}(\bar{D})$ s.t. $(f-u)|_{\partial D} \leq 0$,

we have $(f-u)|_D \leq 0$.

$\overset{\text{Theorem}}{\underset{\text{"SMVT"}}{\iff}}$ $f(p) \leq \frac{1}{2\pi}\int_0^{2\pi} f(p+re^{i\theta})\,d\theta \quad \forall\ \bar{D}(p,r) \subset U.$

sub-mean-value thm.

(Here, as usual, $\mathcal{H}(\bar{D})$ denotes harmonic functions on the closed disk $\bar{D}$, which is to say, functions on $\bar{D}$ which are restrictions to $\bar{D}$ of harmonic function on any open set containing $\bar{D}$.) In this lecture we'll deal with some examples and applications of these functions.

## I. Examples

① Linear combinations: If $f_1, f_2 \in \underline{\mathcal{H}}(U)$, and $c_1, c_2 \in \mathbb{R}_{\geq 0}$, then $c_1 f_1 + c_2 f_2 \in \underline{\mathcal{H}}(U)$.

Proof: Immediate consequence of SMVT. $\square$

In the last lecture we saw that harmonic functions are trivially subharmonic. Now, harmonic functions are smooth ($C^\infty$) — what if we ask for all $C^2$ subharmonic functions?

② $f \in C^2_{\mathbb{R}}(U)$ is subharmonic $\Leftrightarrow \Delta f \geq 0.$

Proof: ($\Longrightarrow$): Let $p \in U$. We have,

for some $G \in C^2_{\mathbb{R}}(U)$ with $\displaystyle\lim_{r \to 0^+} \frac{G(p+re^{i\theta})}{r^2} = 0$,

$$f(p) \underset{\text{SMVT}}{\leq} \frac{1}{2\pi}\int_0^{2\pi} f(p+re^{i\theta})\,d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\Big\{ f(p) + f_x(p)\, r\cos\theta + f_y(p)\, r\sin\theta + f_{xx}(p)\frac{r^2\cos^2\theta}{2} + f_{xy}(p)\, r^2\cos\theta\sin\theta + f_{yy}(p)\frac{r^2\sin^2\theta}{2} + G(p+re^{i\theta})\Big\}\,d\theta$$

$$= f(p) + \frac{r^2}{4}\big(f_{xx}(p) + f_{yy}(p)\big) + \frac{r^2}{2\pi}\int_0^{2\pi}\frac{G(p+re^{i\theta})}{r^2}\,d\theta$$

$$= f(p) + \frac{r^2}{4}\Big\{(\Delta f)(p) + \frac{2}{\pi}\underbrace{\int_0^{2\pi}\frac{G(p+re^{i\theta})}{r^2}\,d\theta}_{\to 0 \text{ as } r \to 0^+}\Big\}$$

Taking $r$ sufficiently small that the $2^{\text{nd}}$ term in braces has smaller absolute value than the first, we see that $(\Delta f)(p) \geq 0$.

$(\Longleftarrow)$: If $f$ has a local maximum at $p$,

then $f_{xx}(p), f_{yy}(p) \leq 0$. So $(\Delta f)(p) > 0 \Longrightarrow$

$f$ cannot have a local maximum at $p$.

So assume $\Delta f > 0$ on $U$. If $h \in \mathcal{H}(\overline{D}(p,r))$

with $f \leq h$ on $\partial D(p,r)$, then $\Delta(f-h) = \Delta f > 0$

$\Longrightarrow$ $f-h$ cannot have a local maximum anywhere on $D$.

Were we to have $f > h$ somewhere in $D(p,r)$,

then $f-h$ attains a (positive) maximum at some

point $q \in D(p,r)$, a contradiction. Conclude

that $f \leq h$ on $D$. So $f \in \underline{\mathcal{H}}(U)$ if $\Delta f > 0$. (*)

Now assume only $\Delta f \geq 0$. Given $\epsilon > 0$,

$\Delta(f + \epsilon |z|^2) > 0$ everywhere on $U$ (**)

(where we used $\Delta |z|^2 = \Delta(x^2+y^2) = 4 > 0$). So

$f + \epsilon |z|^2 \in \underline{\mathcal{H}}(U)$. Given $h \in \mathcal{H}(\overline{D})$ with $h \geq f$

on $\partial D$ $(D = D(p,r))$, let $h_\epsilon := h + \underbrace{\epsilon r^2}_{\text{constant}} \left(\in \mathcal{H}(\overline{D})\right)$

so that $\tilde{h}_\epsilon \geq f + \epsilon|z|^2$ on $\partial D$. By (*) & (**), $f + \epsilon|z|^2 \in \underline{\mathcal{H}}(U)$, and so $\tilde{h}_\epsilon \geq f + \epsilon|z|^2$ on $D$ for all $\epsilon > 0$. Taking $\epsilon \to 0^+$, we find $h \geq f$ on $D$. $\square$

**Example 0** // $f(x,y) = \underbrace{x^2+y^2}_{|z|^2} + C$ is subharmonic. //

**Example 1** // Say $h \in \mathcal{H}(U)$. Then †

$$\Delta h^2 = \partial_x^2 h^2 + \partial_y^2 h^2 = \partial_x 2hh_x + \partial_y 2hh_y$$

$$= 2h_x^2 + 2h_y^2 + 2h\underbrace{\cancel{\Delta h}}_{0} \geq 0$$

$$\Longrightarrow \quad h^2 \in \underline{\mathcal{H}}(U).$$ //

**Example 2** // Say $f \in \underline{\mathcal{H}}(U)$, $f \geq 0$, $f \in C^2_{\mathbb{R}}(U)$.

Let's try the same approach:

$$\Delta f^k = \partial_x^2 f^k + \partial_y^2 f^k = \partial_x k f^{k-1} f_x + \partial_y k f^{k-1} f_y$$

$$= k(k-1)\underbrace{f^{k-2}}_{\geq 0}\underbrace{\{f_x^2 + f_y^2\}}_{\geq 0} + k\underbrace{f^{k-1}}_{\geq 0}\underbrace{\Delta f}_{\geq 0} \geq 0$$

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† writing $\partial_x = \frac{\partial}{\partial x}$, $\partial_y = \frac{\partial}{\partial y}$

$\Longrightarrow f^k \in \underline{\mathcal{H}}(U)$.

So $|z|^{2k}$ is subharmonic. //

(3) (a) $f \in \underline{\mathcal{H}}(U)$, $\varphi : \mathbb{R} \to \mathbb{R}$ nondecreasing & convex on $f(U)$

$\Longrightarrow \varphi \circ f \in \underline{\mathcal{H}}(U)$.

(b) $f \in \mathcal{H}(U)$, $\varphi : \mathbb{R} \to \mathbb{R}$ convex on $f(U)$

$\Longrightarrow \varphi \circ f \in \underline{\mathcal{H}}(U)$.

These are certainly suggested by the the examples above — e.g. $\varphi(x) = x^2$ is convex, and increasing on $\mathbb{R}_{\geq 0}$.

To see how to approach the situation where $\varphi \circ f \notin C^2$, consider

Example 3 // Let $h \in \mathcal{H}(U)$ be given: then

$$|h(p)| \underset{\text{MVT}}{=} \left| \frac{1}{2\pi} \int_0^{2\pi} h(p + re^{i\theta})\, d\theta \right| \leq \frac{1}{2\pi} \int_0^{2\pi} |h(p+re^{i\theta})|\, d\theta$$

$\Longrightarrow |h| \in \underline{\mathcal{H}}(U)$. //

The generalization of the above inequality is given by the

**Lemma:** $f \in C^0_{\mathbb{R}}([a,b])$, $\varphi : \mathbb{R} \to \mathbb{R}$ convex (at least on image$(f)$) $\implies$

$$(\#) \quad \boxed{\varphi\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right) \leq \frac{1}{b-a}\int_a^b (\varphi \circ f)(x)\,dx}.$$

**Proof:** If $\varphi(X) = AX + B$, then

$$\left.\begin{aligned} \text{LHS}(\#) &= \frac{A}{b-a}\int_a^b f(x)\,dx + B \\ \text{RHS}(\#) &= \frac{1}{b-a}\int_a^b (Af(x) + B)\,dx \end{aligned}\right\} \text{ and}$$

are clearly $\boxed{\text{equal}}$ (†)

Now let $\varphi$ be more general, and $(X_0, \varphi(X_0))$ a point on its graph; set $L_\mu(X) = \mu(X - X_0) + \varphi(X_0)$. If we do not have $\varphi \geq L_{\mu_0}$ for any $\mu_0$, then there exists $\mu$ for which there are $X_1 < X_0$ and $X_2 > X_0$ with $\varphi(X_i) < L_\mu(X_i)$ $(i = 1, 2)$, which contradicts the definition of convexity. So $\exists$ such a $\mu_0$; set $L := L_{\mu_0}$.

Specialize to $X_0 := \frac{1}{b-a}\int_a^b f(x)\,dx$. Then

$$\varphi(X_0) = L(X_0) \underset{\text{by (†)}}{=} \frac{1}{b-a}\int_a^b L(f(x))\,dx \leq \frac{1}{b-a}\int_a^b \varphi(f(x))\,dx.$$

$\square$

**Proof of ③(a):** Use SMVT:

$$\varphi(f(p)) \underset{\substack{\Uparrow \\ \varphi \text{ nondecreasing}, \\ f \in \underline{\mathcal{H}}}}{\leq} \varphi\left(\frac{1}{2\pi}\int_0^{2\pi} f(p+re^{i\theta})\,d\theta\right) \underset{\substack{\Uparrow \\ \text{Lemma}}}{\leq} \frac{1}{2\pi}\int_0^{2\pi} \varphi(f(p+re^{i\theta}))\,d\theta$$

$\square$

**Proof of ③(b):**

$$\varphi(f(p)) \underset{\substack{\Uparrow \\ \text{MVT}}}{=} \varphi\left(\frac{1}{2\pi}\int_0^{2\pi} f(p+re^{i\theta})\,d\theta\right) \underset{\substack{\Uparrow \\ \text{Lemma}}}{\leq} \frac{1}{2\pi}\int_0^{2\pi} \varphi(f(p+re^{i\theta}))\,d\theta .$$

$\square$

**Example 4** // $f \in \underline{\mathcal{H}}(U) \Rightarrow e^f \in \underline{\mathcal{H}}(U)$.

If $F \neq 0$ on $U$ is holomorphic, then

$\log|F| \in \mathcal{H}(U) \underset{\exp}{\Longrightarrow} |F| \in \underline{\mathcal{H}}(U)$. //

## II. Maximum principle

**Theorem (MP)** Let $U \subset \mathbb{C}$ be a region (= conn. open set), $f \in \underline{\mathcal{H}}(U)$ a subharmonic function, and $p \in U$ s.t. $f(p) \geq f(z)$ $(\forall z \in U)$. Then $f$ is constant.

**Proof:** Let $K := \{z \in U \mid f(z) = f(p)\}$; $p \in K \Rightarrow K \neq \emptyset$.

- $K$ is open: $f(p) \leq \frac{1}{2\pi}\int_0^{2\pi} f(p+re^{i\theta})\,d\theta \leq \frac{1}{2\pi}\int_0^{2\pi} f(p)\,d\theta = f(p)$
  & $f$ $C^0$ $\Rightarrow f(p+re^{i\theta}) = f(p)$ "$\forall r, \theta$".
- $K$ is closed: given $\{q_i\} \subset K$ sequence w/ limit $q \in U$.
  $f$ $C^0 \Rightarrow f(p) = f(q_i) \longrightarrow f(q)$
  $\Rightarrow f(q) = f(p) \Rightarrow q \in K$.

Therefore $K = U$. $\square$

**Corollary 1** Given: $f \in \underline{\mathcal{H}}(U)$ ; $U_0$ bounded region with $\overline{U}_0 \subset U$ ; $h \in \mathcal{H}(\overline{U}_0)$ with $f \leq h$ on $\partial U_0$. Then $f \leq h$ on $U_0$.

Remark// This goes beyond the definition b/c $U_0$ is more general than a disk. //

**Proof:** $f - h \in \underline{\mathcal{H}}(U_0)$. If $f$ isn't $\leq h$ on $U_0$, then $f-h$ has a maximum on $U_0$ $\Longrightarrow$ $f-h \equiv C > 0$. This contradicts that $f-h \leq 0$ on $\partial U$ and $f-h$ $C^0$. $\square$

$\boxed{\text{Corollary 2}}$ Subharmonicity is a local property.

**Sketch:** Suppose $f \notin \underline{\mathcal{H}}(U)$ but $f$ is subharmonic on every neighborhood of a covering $\{\mathcal{N}_\alpha\}$. Then $\exists\ \bar{D} \subset U$ and $h \in \mathcal{H}(\bar{D})$ such that $f-h \leq 0$ on $\partial D$ and $f-h \not\leq 0$ on $D$

$\Longrightarrow$ $f-h$ has maximum at some $p \in D$

$\Longrightarrow$ $f-h$ has maximum in nbhd. $\mathcal{N}$ of $p$

$\underset{\text{MP} + f \in \underline{\mathcal{H}}(\mathcal{N})}{\Longrightarrow}$ $f-h\big|_{\mathcal{N}} \equiv C$

$\underset{\substack{\text{MP} + f \in \underline{\mathcal{H}}(\text{adjacent } \mathcal{N}_\alpha) \\ + \text{ finiteness of covering of } \bar{D}}}{\Longrightarrow}$ $f-h\big|_D \equiv C\ (>0)$. Contradiction. $\square$

**Example 5** // $f_1, f_2 \in \underline{\mathcal{H}}(U) \Rightarrow f = \max\{f_1, f_2\} \in \underline{\mathcal{H}}(U)$.

**Proof:** Let $\bar{D} \subset U$, $u \in \mathcal{H}(\bar{D})$, $u \geq f$ on $\partial D$.

Then $u \geq f_1, f_2$ on $\partial D$ $\Rightarrow$ $u \geq f_1, f_2$ on $D$

$\Rightarrow$ $u \geq \max\{f_1, f_2\}$ on $D$. $\square$ //

**Example 6** // $f \in \underline{\mathcal{H}}(U)$, $\bar{D} \subset U$, $\tilde{f}(z) := \begin{cases} P_f(z), & z \in \bar{D} \\ f(z), & z \notin D \end{cases}$

$\Longrightarrow \tilde{f} \in \underline{\mathcal{H}}(U)$. [Here $P_f \in \mathcal{H}(D) \cap C^0(\bar{D})$ is given by integrating $f|_{\partial D}$ against the Poisson kernel.]

**Proof:** Clearly $\tilde{f}$ is continuous as $f = P_f$ on $\partial D$ (cf. Lecture 6). $P_f$ isn't defined outside $\bar{D}$ so we can't just "use Example 5". But since $\tilde{f}$ is harmonic on $D$ and subharmonic on $U \setminus D$, we only need to check (by Cor. 2) small disks $\bar{\mathcal{N}} \subset U$ with $\mathcal{N} \cap \partial D \neq \emptyset$.

Suppose (for a contradiction) that $\exists\, h \in \mathcal{H}(\bar{\mathcal{N}})$ s.t. $\tilde{f} - h \leq 0$ on $\partial\mathcal{N}$ but $\tilde{f} - h > 0$ somewhere in $\mathcal{N}$. Now $P_f \geq f$ on $D$ simply because $P_f$ is harmonic on $D$ (& $C^0$ on $\bar{D}$) and $f$ is subharmonic (cf. Lecture 8); it follows that $\tilde{f} \geq f$ everywhere and so $f - h \leq 0$ on $\partial\mathcal{N}$. Again invoking subharmonicity of $f$, we cannot have $\tilde{f} - h > 0$ anywhere in $\mathcal{N} \cap (U \setminus D)$, as $\tilde{f} = f$ there. So $\tilde{f} - h \leq 0$ on $\partial(\mathcal{N} \cap D)$, and $> 0$ somewhere in $\mathcal{N} \cap D$. But $\tilde{f} - h$ is harmonic on $\mathcal{N} \cap D$, and continuous on $\overline{\mathcal{N} \cap D}$, so in view of the maximum principle this is impossible. $\square$

Example 7// $|z|$ is subharmonic $\left( \overset{③(c)}{\Longrightarrow} |z|^k \text{ subharmonic } \forall k \in \mathbb{N} \right)$.

Indeed, we know this away from 0, and by Cor. 2 one only needs to know that no $|z| - h$ ($h$ harmonic) has a max. at 0. Of course, one could then have

$$-h(0) \geq \frac{1}{2\pi} \int_0^{2\pi} \left( |re^{i\theta}| - h(re^{i\theta}) \right) d\theta \underset{\text{MVT}}{=} r - h(0),$$

which is ludicrous. //

Remark// Most definitions of subharmonic functions relax the condition that $f$ be in $C^0_{\mathbb{R}}(U)$, replacing continuity by <u>upper semicontinuity</u> and allow <u>values in $\mathbb{R} \cup \{-\infty\}$</u>. So for example, $\log|z|$ is subharmonic (in this more general sense) on $\mathbb{C}$, because it is actually continuous as a function into $\mathbb{R} \cup \{-\infty\}$, harmonic on $\mathbb{C} \setminus \{0\}$, and satisfies the defining property of subharmonic functions (or equivalently, the "sub-mean-value" property) on a neighborhood of 0. //