Lecture 12: The prime number theorem

I. A Laplace Tauberian theorem

Before resuming our investigation of the counting function $\pi$, we will need an integral analogue of Tauber's theorem (cf. Lecture 8 from 5021).

Recall that the idea of a Tauberian theorem for series was:

If $f(x) = \sum a_n x^n$ has radius of convergence 1 (and is considered to be defined on $D_1$) and

$\lim_{x \to 1^-} f(x)$ exists, then provided some regularity assumption holds for \( \{a_n\} \) (e.g. \( |a_n| \) bounded, \( n a_n \to 0 \), etc.), $\sum a_n$ exists and equals that limit.

The integral analogue is
**Theorem 1**

Let $f : [0, \infty) \rightarrow \mathbb{C}$ be bounded and piecewise $C^0$, and set $U_0 := \{ z \mid \text{Re}(z) > 0 \}$.

Laplace transform

$$g(z) := \int_0^\infty f(t) e^{-zt} \, dt \in \mathcal{H}(U_0).$$

If there exists $\tilde{g} \in \mathcal{H}(U_0)$ extending $g$, then

$$\int_0^\infty f(t) \, dt \text{ exists and equals } \tilde{g}(0).$$

First, why is $g \in \mathcal{H}(U_0)$? This is by the Lemma of Lecture 9, using the uniform convergence of $\int_0^\infty f(t) e^{-zt} \, dt$ on subsets $\overline{U}_e (e > 0)$ and analyticity of the argument in $z$.

By the same token, for any $T > 0$ the function

$$g_T(z) := \int_0^T f(t) e^{-zt} \, dt$$

is entire.

Now since $\tilde{g} \in \mathcal{H}(U_0)$, it is holomorphic on an open set containing $\overline{U}_0$; call this $\tilde{U}_0$. Given any $R > 0$, compactness of $[-R, R] \Rightarrow$
$$\Delta_R := \min_{y \in [-R, R]} \left( \max \{ r | D(iy, r) \subset \tilde{U}_0 \} \right) > 0.$$ 

Taking $\delta \in (0, \Delta_R)$, $\tilde{g}$ is defined on a region containing $iR$.

\[ \gamma := \gamma_+ + \gamma_- \]

and the region it encloses. By Cauchy's Theorem \((e^{-z} (1 + \frac{e^2}{R^2}) = 1)\),

\[ \tilde{g}(0) - g_T(0) = \frac{1}{2\pi i} \int (\tilde{g}(z) - g_T(z)) e^{Tz} \left(1 + \frac{e^2}{R^2} \right) \frac{dz}{z} \]

\[ = \frac{1}{2\pi i} \int_{\gamma} H_T(z) \, dz. \]

Next, recall that \(\|f\|_{(0, z_0)} \leq B\) by assumption.

**Lemma 1:** \( \left| \frac{1}{2\pi i} \int_{\gamma} H_T(z) \, dz \right| \leq \frac{B}{R} \).

**Proof:** For \( z \in U_0 \), \( |g(z) - g_T(z)| = \left| \int_T^{z_0} f(t) e^{-e^t} dt \right| \leq B \int_T^{z_0} |e^{-e^t}| dt = \frac{B}{R} \frac{e^{-Re(z)T}}{e^{-Re(z)T}}. \)

For \( |H| = R \), \( e^{T \left(1 + \frac{e^2}{R^2} \right) \frac{1}{z}} = \frac{e^{Re(z)T}}{R} \cdot \left| \frac{R + \frac{e^2}{R}}{z} \right| \).
\[ e^{\text{Re}(z) T} \left| \frac{\text{Re}(z)}{R^2} \right| \uparrow \] 

\[ \frac{R^2 e^{i\theta}}{R \text{Re} \theta} = e^{i\theta + i \text{Re}(z)} \]

So for \( z \in \gamma_+ \),

\[ |H_T(z)| \leq \frac{B}{\text{Re}(z)} e^{-\text{Re}(z) T} \frac{2 \text{Re}(z)}{R^2} = \frac{2B}{R^2} \]

and

\[ \frac{1}{2\pi} \left| \int_{\gamma_+} H_T(z) e^{iz} \text{d}z \right| \leq \frac{1}{2\pi} L(\gamma_+) \frac{2B}{R^2} = \frac{B}{R} \]

Lemma 2: \[ \left| \frac{1}{2\pi i} \int_{\gamma_-} g_T(z) e^{Tz} \left(1 + \frac{z^2}{R^2}\right) \frac{dz}{z} \right| \leq \frac{B}{R} \]

Proof: Write \( C_- \) for \( \{ |z| = R \} \cap \{ \text{Re}(z) \leq 0 \} \), so that \( \gamma_- - C_- \) is closed. As the integrand only has a pole at \( z = 0 \),

\[ \int_{\gamma_- - C_-} g_T(z) e^{Tz} \left(1 + \frac{z^2}{R^2}\right) \frac{dz}{z} = 0. \]

For \( z \in C_- \),

\[ |g_T(z)| = \left| \int_0^T f(t) e^{-\text{Re}(z) t} \text{d}t \right| \leq B \int_0^T e^{-\text{Re}(z) t} \text{d}t = B \left\{ \frac{1}{\text{Re}(z)} - \frac{e^{-\text{Re}(z) T}}{\text{Re}(z)} \right\} \]

and so

\[ \frac{1}{2\pi} \left| \int_{C_-} g_T(z) e^{Tz} \left(1 + \frac{z^2}{R^2}\right) \frac{dz}{z} \right| \leq \frac{1}{2\pi} B \left\{ \frac{1}{\text{Re}(z)} - \frac{e^{-\text{Re}(z) T}}{\text{Re}(z)} \right\} \]

\[ \frac{R}{2\pi} e^{-\text{Re}(z) T} \frac{2 \text{Re}(z)}{R^2} = \frac{B}{R} \]

Lemma 3: \[ \lim_{T \to \infty} \int_{\gamma_-} \tilde{g}_T(z) e^{Tz} \left(1 + \frac{z^2}{R^2}\right) \frac{dz}{z} = 0. \]

Proof: Set \( h(z) := \tilde{g}_T(z) \left(1 + \frac{z^2}{R^2}\right) \frac{z}{z} \) so the integral is
\[ \int_{\gamma_{-}} h(x) e^{Tz} \, dx, \] where \( h(x) \) is analytic on a neighborhood of \( \gamma_{-} \). By compactness of \( \gamma_{-} \), \( \| h \|_{\gamma_{-}} \leq M \), and so on \( \gamma_{-} \) we have \( |h(x) e^{Tz}| \leq Me^{RzT} \).

Substituting \( \gamma_{-} = \gamma_{0}^{1} + \gamma_{0}^{1} \) (see figure), we have (uniformly) \( \| h(x) e^{Tz} \|_{\gamma_{0}} \leq Me^{-T\delta} \rightarrow 0 \), so that the \( \int_{\gamma_{0}} \) part of the integral does limit to 0. On the remaining small part \( \gamma_{-}^{1} \), the integrand is bounded by \( M \).

So given \( \epsilon > 0 \), if we take \( \delta_{1} < \frac{\epsilon}{8M} \) and

\[ T > \frac{1}{\delta_{1}} \log \left( \frac{2ML(\gamma_{-})}{\epsilon} \right), \] then

\[ \left| \int_{\gamma_{-}} h(x) e^{Tz} \, dx \right| \leq \int_{\gamma_{0}} Me^{-T\delta_{1}} \, dx + \int_{\gamma_{-}^{1}} M \, dx \leq L(\gamma_{-}) Me^{-\frac{\epsilon}{8M}} + \frac{\epsilon}{8M} \leq \frac{\epsilon}{\epsilon} = \epsilon. \]

**Proof of Theorem 1:** Given \( \epsilon > 0 \), pick \( R \) s.t. \( \frac{B}{R} < \frac{\epsilon}{2} \)

and (by Lemma 3) \( T \) s.t. \( \left| \int_{\gamma_{-}} h(x) e^{Tz} (1 + \frac{Rz}{R}) \, dx \right| < \frac{\epsilon}{2} \).

Then

\[ \left| g(0) - g_{T}(0) \right| = \left| \frac{1}{2\pi i} \int_{\gamma_{-}} H_{T}(z) \, dz \right| \leq \frac{B}{R} + \frac{b}{R} + \frac{\epsilon}{2} < \epsilon, \]

which proves that

\[ g(0) = g_{T}(0), \]

Lemma 1 Lemma 2
\[ \lim_{T \to \infty} g_T(0) \] exists and equals \( g(0) \).

But also, \( \lim_{T \to \infty} g_T(0) = \lim_{T \to \infty} \int_0^T f(t) e^{-it} dt \) (by def'n.)
\[ = \int_0^\infty f(t) dt. \]

II. Proof of the prime number theorem

Back to our "counting functions". Recall that
\[ \varphi(x) := \sum_{p \leq x} \log p \]
and
\[ \Phi(x) := \sum_{p \leq x} \frac{\varphi(p)}{p} \]
extends to a meromorphic function on \( \mathbb{C} \) with poles at \( s = 1 \) (with residue 1) and at the zeros of the Riemann zeta function (which only occur, if at all, on \( \frac{1}{2} < \text{Re}(s) < 1 \) in this region).

**Theorem 2** \[ \varphi(x) = O(x) \] (that is, \( \exists \, C > 0 \) and \( M > 0 \) s.t. \(|\varphi(x)| \leq Cx \forall \, x \geq M \).
Proof: For \( n \in \mathbb{N} \),

\[
2^{2n} = (1+1)^{2n} = \sum_{j=0}^{2n} \binom{2n}{j} \geq (2^n) = \prod_{p \text{ prime}, \; n \leq p \leq 2n} p = e^{\phi(2n) - \phi(n)}
\]

\[
\Rightarrow \quad \phi(2n) - \phi(n) \leq 2n \log 2
\]

\[
\Rightarrow \quad \phi(2^k) = \sum_{k=1}^{\frac{1}{2}} \left( \phi(2^k) - \phi(2^{k-1}) \right) \quad \left\{ \phi(1) = 0 \right\}
\]

\[
\leq \sum_{k=1}^{\frac{1}{2}} 2^k \log 2
\]

\[
< 2^{k+1} \log 2.
\]

Given \( x > 1 \), \( 2^{-k} < x < 2^k \) for some \( k \) and

\[
\phi(x) < \phi(2^k) < 2^{k+1} \log 2 < 2^{1-\frac{1}{2}} \cdot x.
\]

We will also need the following consequence of Theorems 1 & 2:

**Proposition** \[
\int_{1}^{\infty} \frac{\phi(x) - x}{x^2} \, dx \text{ converges!}
\]

**Proof:** Set \( f(x) = \phi(e^x) e^{-x} - 1 \) (piecewise \( C^0 \) on \([0, \infty)\)). By Theorem 2, \( |\phi(e^x) e^{-x}| < Ce^x e^{-x} = C \) on \([0, \infty)\). So Theorem 1 applies, and we have that...
\[
\int_0^\infty f(t) \, dt = \int_1^\infty \frac{\varphi(x) - x}{x} \, dx \quad \text{converges provided}
\]
\[
\left( \frac{x = e^t}{\log x = t} \right) \quad \text{that the Laplace transform}
\]
\[
g(t) = \int_0^\infty f(t) e^{-2t} \, dt \in \mathcal{H}(U_0) \quad \text{extends to}
\]
\[
g \in \mathcal{H}(\overline{U_0}). \quad \text{For \text{Re}(t) > 0},
\]
\[
g(t) = \int_0^\infty \frac{\varphi(e^t) - e^t}{e^t} e^{-2t} \, dt
\]
\[
= \int_0^\infty \frac{\varphi(e^t) - e^t}{e^{2t}} e^{-2t} e^t \, dt
\]
\[
= \int_0^\infty \frac{\varphi(x) - x}{x^{2+2}} \, dx
\]
\[
= \Phi(z+1) - \frac{1}{z+1}
\]

which extends to a meromorphic function on \text{Re}(z) > -\frac{1}{2}
with no poles on the line \text{Re}(z) = 0. \quad \text{(This is Theorem 3 of Lecture 11, with} z = 2+1). \quad \text{So our}
\]
\[
\Omega \text{ is just } \{\text{Re}(z) > -\frac{1}{2} \} \setminus \text{poles (} \supset \overline{U_0}\), \quad \text{and we}
\]
\[
\text{have met the conditions of Theorem 1.} \quad \square
\]

**Theorem 3** \[ \varphi(x) \sim x, \text{ i.e. } \lim_{x \to \infty} \frac{\varphi(x)}{x} = 1. \]**
Proof: We must show \( \limsup_{x \to \infty} \frac{\varphi(x)}{x} \leq 1 \) and \( \liminf_{x \to \infty} \frac{\varphi(x)}{x} \geq 1 \). If the first fails, let \( \delta > 0 \) such for certain arbitrarily large numbers \( y \), \( \varphi(y) > (1 + 2\delta)y \), hence \( \varphi(y) > (1 + 2\delta)y > (1 + \delta)x \) for \( y < \frac{x}{\psi(y)} \), where \( \psi := \frac{1 + 2\delta}{1 + \delta} \). But then
\[
\int_y^{\psi y} \frac{\varphi(x) - x}{x^2} \, dx > \int_y^{\psi y} \frac{\delta x}{x} \, dx = \delta \log(\psi)
\]
for those same numbers \( y \), contradicting the existence of the integral in the Proposition.

[Exercise: Show that we cannot have \( \liminf_{x \to \infty} \frac{\varphi(x)}{x} < 1 \) by the same approach, by considering intervals \( \theta y < x = y \) with \( \theta < 1 \), where \( \varphi(x) < (1 + \delta)x \).]

This completes the proof.

Now let
\[
\pi(x) := \text{number of primes } \leq x
\]
Prime Number Theorem \[ \Pi(x) \sim \frac{x}{\log(x)} \]

Proof: We have

\[ \phi(x) = \sum_{\substack{p \leq x \text{ prime}}} \log p \leq \sum_{p \leq x} \log x = \Pi(x) \log x \]

\[ \Rightarrow \Pi(x) \cdot \frac{\log x}{x} \geq \frac{\phi(x)}{x} \quad (**) \]

Further, for \( 1 < y < x \),

\[ \Pi(x) = \Pi(y) + \sum_{y < p \leq x} 1 \leq \Pi(y) + \sum_{y < p \leq x} \frac{\log p}{\log y} \]

\[ \Rightarrow \Pi(x) \leq y + \frac{\phi(x)}{\log y} \quad (***) \]

Taking \( y = \frac{x}{\log^2 x} \), then (***) gives

\[ \Pi(x) \frac{\log x}{x} \leq \left( \frac{x}{\log^2 x} + \frac{\phi(x)}{\log x - 2 \log(\log y)} \right) \frac{\log x}{x} \]

\[ = \frac{1}{\log x} + \frac{\phi(x)}{x} \cdot \frac{\log x}{\log x - 2 \log(\log x)} \]

Which together with (**) gives
\[
\phi(x) \leq \pi(x) \frac{\log x}{x} \leq \frac{1}{\log x} + \frac{\phi(x)}{x} \frac{\log x}{\log x - 2 \log(\log x)}.
\]

whereupon Theorem 3 applies to give the result.

**Remark** The function

\[\text{Li}(x) := \int_0^x \frac{dt}{\log t}\]

is actually known to do a better job at approximating \(\pi(x)\) than \(\frac{x}{\log x}\). By a result of Schoenfield, if the Riemann hypothesis holds then one has

\[|\pi(x) - \text{Li}(x)| \leq \frac{1}{8\pi} \sqrt{x} \log x \quad \text{for all } x \geq 2657.\]