Lecture 18: The Picard theorems

I. The monodromy theorem

A function element is a pair \((f, U)\), \(U\) a region in \(\mathbb{P}^1\) and \(f \in \text{Hol}(U)\). Two elements \((f, U)\) and \((g, V)\) are said to be \textbf{direct analytic continuations} of each other if \(U \cap V \neq \emptyset\) and \(f|_{U \cap V} = g|_{U \cap V}\). This generates an equivalence relation,

\[
(f, U) \equiv (g, V) \iff \text{def. they are connected by a chain of direct analytic continuations.}
\]

An equivalence class of function elements is called a \textbf{global analytic function}. Very roughly speaking, the point of view taken on Riemann surfaces in Chapter 8 of Ahlfors is that they are the "graphs" of these global analytic functions (obviously multivalued), "completed at branch points". The one-to-one identification (Ahlfors, Ch.8 Thm.9) of algebraic curves

\[
\{ (w, z) \mid P(w, z) = 0 \} \subset \mathbb{C} \times \mathbb{C}
\]

and global analytic functions with finitely many sheets/branches and algebraic singularities is almost exactly the result on connectedness of algebraic curves we proved last term with
Rouché's theorem.

The upshot is that "global analytic functions" give one notion of analytic continuation. In this section we review a more elementary one. Let $z_0 \in \mathbb{C}$, and take $f$ to be a germ of an analytic function at $z_0$, i.e., a convergent power series centered there. Consider

$$Y : [a, b] \rightarrow \mathbb{C} \quad \text{[C^0 path with } Y(a) = z_0]$$

and choose

$$\{ T = \{ a_0, a_1, \ldots, a_{n+1} \} \text{ a partition of } [a, b]$$

$$a < \ldots < b$$

$$\{ \{ D_i \}_i \text{ disks with } D_i \supset Y([a_i, a_{i+1}]) \}. \tag{\#}$$

If there exist $f_i \in \mathbb{H}(D_i)$ with $f_i \big|_{D_i \cap D_{i+1}} = f_{i+1} \big|_{D_i \cap D_{i+1}} \,(*)$ for any choice $(\#)$, we say that $f$ admits analytic continuation along $Y$ and call the power series of $f_n$ at $z(b)$ the "terminal germ". (Denote this by $g$ for now.)

**Proposition.** The terminal germ is independent of the choices $(\#)$.

**Proof.** We need only check that we can choose $T$ and the $\{ D_i \}$ "independently" without affecting $g$. If we pick different $D_i$, then the "common refinement" $D_i := D_i \cap D_j$.
Still produces a continuation, and it's clear that it is the restriction of the other two. (Crucial here is that the $D_i$ are, while not disks, connected.) If we add some $c \in (a_k, a_{k+1})$ to $T_i$, then we may choose $(f_k, D_k)$ for both subintervals $[a_k, c]$ and $[c, a_{k+1}]$. Done.

**Theorem 1 ("The monodromy theorem")**

Let $\mathcal{D} \subset \mathbb{C}$ be a connected open set; and take $f$ to be an analytic germ at $z_0 \in \mathcal{D}$, admitting analytic continuation along any path in $\mathcal{D}$. Then given any two paths $y, y' : \mathbb{T} \to \mathcal{D}$, homotopic in $\mathcal{D}$, the terminal germs $f_y$ and $f_{y'}$ are equal.

**Proof:** Let $\psi : [a, b] \to [0, 1] \to \mathcal{D}$ be a homotopy (with $\psi(a) = z_0$, $\psi(b) = y$, and $\psi(0, 0) = y$, $\psi(1, 1) = y')$. From last term, using the uniform continuity of $\psi$ (and the finite distance from image $[\psi]$ to $\mathcal{D}$), there exists a partition

$$0 = \delta_0 < \delta_1 < \ldots < \delta_m = 1$$

such that successive $\psi(\tau, \delta_j)$ are "close together": that is, there exists a choice of the $a_i$ and $D_i \subset \mathcal{D}$ such that $\psi([a_i, a_{i+1}], \delta_j)$ and $\psi([a_i, a_{i+1}], \delta_{j+1})$ are contained...
Example // The dilogarithm

\[ L_2(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^2} \]

exhibits interesting "monodromy" (= continuation properties, often around a "singular point" through which continuation is impossible). Confirmed once around \( z = 1 \), it becomes

\[ L_2(z) + 2\pi i \log(z), \]

which is no longer analytic at \( 0 \)!! If \( Y \) denotes the commutator \( Y_1^{-1} Y_0^{-1} Y_1 Y_0 \)

then the continuation along \( Y \) goes

\[ L_2(z) \xrightarrow{Y_0} L_2(z) \xrightarrow{Y_1} L_2(z) + 2\pi i \log(z) \]

\[ \xrightarrow{Y_0^{-1}} L_2(z) + 2\pi i (\log(z) - 2\pi i) \]

\[ \xrightarrow{Y_1^{-1}} L_2(z) - 2\pi i \log(z) + 2\pi i (\log(z) - 2\pi i) = L_2(z) + 4\pi^2. \]

This path is homologous but not homotopic to zero in \( C(\{0,1\}^1) = \mathbb{R} \), which emphasizes the importance of "homotopy" in the Theorem.
II. Little Picard (bis)

We now use the monodromy theorem and our product of our discussion of modular forms (the \( \lambda \) function) to give a more natural proof of this result, done last term with Nevanlinna theory.

**Theorem 2** \( f \in \text{Hol}(\mathbb{C}) \) with \( f(\mathbb{C}) \subseteq \mathbb{C} \setminus \{g,a,b\} \) is constant.

**Proof:** Replacing \( f \) by \( \frac{f(z) - a}{b - c} \), it suffices to do the case \( a = 1, b = 0 \), so that \( f(\mathbb{C}) \subseteq \mathbb{C} \setminus \{0\} \), which happens to be the range of the \( \lambda \)-function! There exists \( \lambda \in \text{Hol}_1 \) with \( \lambda(\tau_0) = f(\tau_0) \); and since \( \lambda \) is 1-1 on \( \Delta_2 \), \( \lambda'(\tau_0) \neq 0 \). So there is a local inverse \( \lambda_0^{-1} : \Delta_0 \rightarrow \mathbb{C} \), when \( \Delta_0 \) is a small disk. Take \( \lambda_0 \) to be the component of \( f^{-1}(\Delta_0) \) containing 0, and set

\[ h := \lambda_0^{-1} \circ f : \Delta_0 \rightarrow \lambda_0^{-1}(\Delta_0) \subseteq \mathbb{C} \]

In a diagram:

\[
\begin{array}{ccc}
\mathbb{C} & \xrightarrow{f} & \mathbb{C} \setminus \{0,1\} = \Delta_0 & \xrightarrow{\lambda} & \mathbb{C} \\
\mathbb{C}^{-1} & \xrightarrow{\lambda^{-1}} & \lambda^{-1}(\Delta_0) & \xrightarrow{\lambda_0} & \Delta_0 \end{array}
\]
We need something stronger than "h can be analytically continued along all paths in \( \mathbb{C} \)" — we want this AND that all analytic continuations remain within \( \mathbb{H} \).

Suppose otherwise: then there exists a path \( \gamma \)

where analytic continuation is not possible past \( z_1 \), OR \( \text{Im}(h) \) becomes non-positive there. Now, \( \exists \, \tau_1, \epsilon \) s.t. \( \lambda(\tau_1) = f(z_1) \), and repeating the above construction gives \( \Delta_i, \lambda^{-1}, \tilde{\gamma}_i, \text{ and } \gamma_i \) of \( : \mathbb{H} \rightarrow \mathbb{H} \). Denoting the analytic continuation of \( h \) (to \( z_2 \)) by \( h \), we have (as \( z_2 \) "varies to the left of \( z_1 \)"") in \( \gamma \cap \Delta_i \)

\[
\lambda((\tilde{\gamma}_i(z_2)) = f(z_2) = \lambda(h(z_2)).
\]

Hence, for some \( \gamma \in \Gamma(2) \), we have in a neighborhood of \( z_2 \)

\[
h_1 := \gamma \circ \tilde{\gamma}_i = h.
\]

(A priori, we could have a different \( \gamma \) for each \( z_2 \), s.t. \( \gamma(\tilde{\gamma}_i(z_2)) = h(z_2) \); but \( \tilde{\gamma}_i \) and \( h \) vary continuously and \( \gamma \) cannot.) But then \( h_1 \) continues \( h \) past \( z_1 \) with \( \text{Im}(h) > 0 \), a contradiction.

Therefore \( h \) continues along every path in \( \mathbb{C} \),
with positive imaginary part. By the Monodromy Theorem and simple-connectedness of $C$, it continues to an entire function

$$H : C \rightarrow \mathbb{H}.$$ 

But then $e^{iH}$ is bounded, hence (by Liouville) constant. So $H$ hence $f \circ \lambda \circ H$ is constant.
III. Montel's normality criterion

This will be required for the proof of "big Picard".

Theorem 3: Let \( S \subset \mathbb{C} \) be a connected open set, and let \( a, b \in \mathbb{C} \) distinct. Define \( \mathcal{F} := \{ f \in \mathcal{H}(S) \mid f(S) \subset \mathbb{C} \setminus \{a, b\} \} \).

Then \( \mathcal{F} \) is a normal family "in the classical sense": any sequence \( \{f_n\} \subset \mathcal{F} \) has a subsequence which either converges normally or tends "to the constant \( 0 \)" uniformly on compact subsets.

Remarks: (a) As above, we may assume \( a = 1, b = 0 \).

(b) We may assume \( S = D_1 \). (Normality is a local property: any \( S \) is covered with a countable collection of balls, and if any \( \{f_n\} \subset \mathcal{F} \) has a subsequence converging on \( D_1 \) normally in this sense, a subsubsequence converges normally on \( D_1 \), \( \ldots \), etc., then the diagonal subsequence converges normally on all disks, and any compact \( K \) can be written as a union of finitely many compact \( K_i \subset D_i \).)

(c) We may replace \( \mathcal{F} \) by \( \mathcal{F}^1 := \{ f \in \mathcal{F} \mid |f(0)| < 1 \} \). (Given \( \{f_n\} \subset \mathcal{F}^1 \), either a subsequence lies in \( \mathcal{F}^1 \) or in \( \sqrt{f_n} \). Suppose we are in the latter setting; i.e., \( \sqrt{f_{n_k}} \in \mathcal{F}^1 \) and we have proved the result for \( \mathcal{F}^1 \).)
Since the $f_{n,k}$ don't have 0 in their range, $1/f_{n,k}$ are holomorphic, and by Hurwitz, a subsequence of the $f_{n,k}$ either go normally to 0, a non-zero-torsion holomorphic function, or so. Therefore the same applies to $1/f_{n,k}$.

(a) If $\mathcal{A}$ is a set of functions, holomorphic on $D_1$, with range in $\mathbb{C}$, then $\mathcal{A}$ is normal in the compact sense. This was a HW problem, but I give the proof anyway.

Given $\{g_n\} \subset \mathcal{A}$, $|e^{i\gamma_n}| < 1 \Rightarrow e^{i\gamma_n}$ holomorphic and uniformly bounded.

$I$ normally convergent subsequence $e^{i\gamma_{n_k}}$ (with uniform limit 0 or $e^{i\alpha}$ (uniform 0) by Hurwitz.

(Here $g$ exists by $D$ is simply connected.) In the first case, $g_{n_k} \xrightarrow{\text{uniformly}} e^{i\alpha}$; in the second, there are actually 2 possibilities. Let $K$ be compact; then $e^{ig(K)} \subset \text{compact annulus} \Rightarrow g(K)$ is contained in a compact rectangle in $\mathbb{C}$. One has from the uniform convergence $e^{i\gamma_{n_k}1/k} - e^{i\gamma_{1/k}}$ that $g_{n_k}(x)$ goes uniformly to $g(x)$ mod $2\pi i$.

From this, either $g_{n_k}(x)$ has a subsequence $\xrightarrow{\text{uniformly}} g_K + m$ or a subsequence with $\pm \text{Re}(g_{n_k}) \xrightarrow{\text{uniformly}} \infty$.

Proof of Theorem 3: So assume

$$\overline{F}_1 := \{ f \in \text{Hol}(D_1) \mid f(D_1) \subset \mathbb{C} \setminus \{0, 1\}, |f(0)| \leq 1 \}.$$
We need to produce a (classically) normally convergent subsequence of a given \( f_n \) \( \subset F_i \). Since \( \{ f_n(0) \} \subset D_1 \), we have \( f_n(0) \to 0 \) as \( n \to \infty \).

**Case 1** \( (\mu \neq 0,1) \)

Fixing a branch of \( \lambda^{-1} \) in a neighborhood of \( \mu \), define \( \hat{f}_n = \text{analytic continuation of} \lambda^{-1} f_n \) (a priori defined in orb. of 0) as in \( \mathbb{III} \). Since \( \hat{f}_n : D_1 \to \mathbb{C} \), by Remark (8) \( \hat{f}_n \) converging normally to a (necessarily analytic, nonzero) limit function \( g \). A priori, \( g(D_1) \subset \overline{\mathbb{C}} \); but since \( g(0) = \lambda^{-1}(\mu) \in \mathbb{C} \) and \( D_1 \) is open, the open mapping theorem \( g(D_1) \subset \mathbb{C} \Rightarrow g \) is defined, and \( f_{n,k} = \lambda \cdot \hat{f}_{n,k} \to \lambda g \).

**Case 2** \( (\mu = 1) \)

Let \( h_n = \sqrt{f_n} \) be chosen so that \( h_n(0) \to 0 \) as \( n \to \infty \). (We can take \( h_n \) as simply connected, and \( f_n \) is nowhere 0.) Then \( h_n \) omits 0, 1, \( |h_n(0)| \leq 1 \), \( h_n(D_1) \subset \{ |z| < 1 \} \), so we're back in Case 1. Squaring the normally convergent \( h_{n,k} \Rightarrow f_{n,k} \) conv. normally.

\( \uparrow \) it can't be \( \equiv 0 \) or \( \equiv \infty \) b/c \( \mu \in \overline{D_1}^* \).
\textbf{Case 3 (x = 0)}

Let \( \hat{r}_0 = 1 - r_{nf} \); then in Case 2.

This completes the proof.

One can see this Montel theorem as an example of the heuristic (and in general false)

"Bloch principle" that properties which force entire functions to be constant also force families of functions to be normal.
IV. Big Picard

**Theorem 4** Let $U \subset C$ be a connected open set containing $z_0$, and let $f$ and $(U \setminus \{z_0\})$ have an essential singularity at $z_0$. Then $f$ omits at most one value $\alpha \in C$.

**Corollary** In a neighborhood of an essential singularity, an analytic function takes every complex value, with possibly one exception, infinitely often.

**Proof of Corollary (assuming Theorem):** Take $z_1 \in U$ where $f(z_1) = b \neq a$. (This exists by the Theorem.) Then there a disk about $z_0$ excluding $z_1$; applying Theorem 4 again, $f(z_2)$ in that disk where $f(z_2) = b$; and so on.

Note that the Corollary strengthens Casorati-Weierstrass, which merely asserted that the range of $f$ was dense in $C$. The Theorem also implies Little Picard, since in that result the case of an essential singularity
at \( a \) is the only nontrivial one. (Otherwise we can use the fundamental theorem of algebra, since \( f \) is holomorphic and has a pole or \( \infty \) is a polynomial.)

**Proof of Theorem 4:** Wolog assume \( U = D_1 \), \( z_0 = 0 \), \( f \) omits \( a, b \in \mathbb{C} \), & show \( 0 \) is a pole or removable singularity of \( f \).

Suppose \( 0 \) is not a pole: i.e., \( \exists M \) and \( z_n \to 0 \) s.t. \( |f(z_n)| \leq M \) (Wlog). Then exists a subsequence \( (z_{n_k})_{k \to \infty} \to w \in \mathbb{C} \). We may assume \( |z_{n_k}| \) strictly decreasing and that \( |z_{n_k}| < \frac{1}{2} \) (Wlog).

Set \( \lambda_k := 2z_{n_k} \), and consider the sequence of functions \( f_k : D_1^+ \to \mathbb{C} \setminus \{a, b\} \) defined by \( f_k(z) := f(\lambda_k z) \). By Theorem 3, passing to a subsequence if necessary, \( f_k \) converges uniformly to a holomorphic function on \( D_1 \). In fact, it can't converge to \( 0 \), because \( f_k'(\frac{1}{2}) = f'(\lambda_{n_k}) \nrightarrow 0 \).

Write \( F \) for the limit function.

Now put \( K_k := \max \{|f_k(z)| : |z| = \frac{1}{2}\} \) (\( \max \{|f(z)| : |z| = |\lambda_{n_k}|\} \)). Since \( f_k \) unif. \( \to F \) on the compact set \( |z| = \frac{1}{2} \),
and more $|F_n(t)| < \infty$, we get $K := \sup_{k \in \mathbb{N}} K_k < \infty$.

By the MNP, however, we have

$$\|f(t)\|_{L^\infty} \leq \max \{ K_{k-1}, K_k \} \leq K$$

where $A(\mathbb{Z}_n)$ is a ring of integers

$$|2_{z_n} - |z| \in \{ \pm 1, \pm 3, \pm 5, \ldots \}$$

$$\Rightarrow \quad \|f(z)\|_{L^\infty} \leq K$$

$$\Rightarrow \quad 0 \text{ is a removable singularity, }$$

\text{done.}