Lecture 19: The Bloch and Landau constants

The next few lectures will be devoted to a topic known as geometric function theory, which has interesting connections with the topics studied so far this term: normal families, analytic continuation, Riemann Mapping Thm., etc.

To get a feeling for the flavor of this area, consider $f \in \mathcal{H}(D)$ ($D := D_1$) with $f(0) = 0$ and $f(1) = 1$. We might ask:

1. Is there a “minimal” subdisk of the domain or range where $f$ must be injective (i.e., 1-1)?

2. If $f$ is injective on all of $D$, are there upper bounds on the power-series coefficients? (Note that $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$.)

(bonus) In both cases, what are the “extremal functions” like?

The answer to the second question, as we shall see, is “yes” simply because the power-series coefficients are continuous functions on a compact set (of functions).
in the topology of normal convergence. That the actual best bounds are \[ \text{known as the Bieberbach Conjecture} \]
(1916) proved by L. de Branges (1985). That the family of schlicht functions is normal (or rather, its proof) will become part of the proof of the Nonexistence Theorem for Riemann surfaces, which we shall do at the end of this segment of the course.

The answer to the first question is also yes, which we will prove below, but the “best bounds” are only conjectured. I should point out that A. Baernstein, a professor in our department until his death in 2014, made groundbreaking contributions to both problems.
I. Bounded functions

Let's first restrict to a case of the first problem where the family is obviously normal hence solutions to extremal problems guaranteed.

Set \( \mathcal{F} := \{ f \in \mathcal{F}(1) \mid f(0) = 0, f'(0) = 1 \} \),
\( \mathcal{F}_\alpha := \{ f \in \mathcal{F}(\alpha) \mid f(0) = 0, f'(0) = 1 \} \).

**Lemma 1:** \( f \in \mathcal{F}_\alpha, \|f\| \leq M \Rightarrow M \geq 1, f(0) \geq \frac{D}{6M} \).

**Proof:** By the Cauchy inequalities,

\[
|a_n| \leq \lim_{n \to 1^-} \frac{\|f\|_{\mathcal{D}(r)}}{r^n} \leq M \quad (\forall n)
\]

\[
\Rightarrow M \geq 1.
\]

For \( z \in \mathcal{D}_{\alpha}^{1/4m} \), we have

\[
|f(z)| \geq |z| - \varepsilon \sum_{n \geq 1} |a_n||z|^n
\]

\[
\geq \frac{1}{4M} - \frac{\varepsilon}{M} \sum_{n \geq 2} \frac{M}{(4M)^n} \geq \frac{1}{4M} \left( \frac{(4M-1)-M}{M(4M-1)} \right) \geq \frac{1/6M}{1}.
\]

\[
\frac{M(4m)^2}{1-\frac{1}{4m}} = \frac{1}{4(4m-1)} \geq \frac{2}{3}
\]

\[
\frac{3m-1}{4m-1} \geq \frac{2}{3}
\]
Set \( g(z) := f(z) - w \) for any \( w \in D_{1/6M} \). Then

\[
|f(z) - g(z)| = |w| < \frac{1}{6M} \leq |f(z)|
\]

\[\Rightarrow\] \( f \) and \( g \) have the same \# of zeros in \( D_{1/4M} \)

\[f(0) = 0\]

\[\Rightarrow\] \( g(0) = 0 \) for some \( z_0 \in D_{1/4M} \)

\[\Rightarrow\] \( w \in f(D_{1/4M}) \subset f(D_{1/6M}) \). □

Rescaling gives

**Proposition 1** \( g \in \mathfrak{K}(D_R) \), \( g(0) = 0 \), \( \log'(0) = \mu > 0 \), and \( \|g\|_{D_R} \leq M \) \(\Rightarrow\) \( g(D_R) > D_{R^{\mu^2/6M}} \).

**Proof:** We have \( f(z) := \frac{g(Rz)}{R \log'(0)} \in \mathfrak{K} \), with \( \|f\|_{D} \leq \frac{M}{\mu R} \).

By Lemma 1, \( f(0) = D_{R^{\mu^2/6M}} \); the conclusion follows. □

We will also need

**Lemma 2:** \( f \in \mathfrak{K}(D(x, r)) \) s.t.

\[
(\#) \quad |f'(z) - f'(x)| < |f'(z)| \quad (4 \geq x \in D^*(x, r))
\]

\[\Rightarrow\] \( f \) is injective.
Proof: Given $Y = \overline{[z_1, z_2]} \subset D(x, r)$,

$$|f(x) - f(y)| = \left| \int_Y f'(z) \, dz \right|$$

FTC

$$\geq \left| \int_Y f'(z) \, dz \right| - \left| \int_Y \{f'(z) - f'(y)\} \, dz \right|$$

$\Delta$ inequality

$$\geq |f(c)| |z_1 - z_2| - \int_Y [f'(z) - f'(y)] \, dz$$

$$> 0$$

by (**)

$$\implies f(x_1) \neq f(x_2).$$
II. Bloch's† Theorem

Given $f \in \mathfrak{f}$ with no bound, we would like to find some disk (not necessarily centered about zero) in the domain $\mathfrak{f}$/range so that $f$ is injective there.

Start by assuming $f \in \overline{\mathfrak{f}}$. Put (for $r \in [0,1]$)

$$K(r) := \|f'\|_{D_r} = \|f'\|_{\overline{D}_r} ;$$

$$h(r) = (1-r) \cdot K(r) \in C^0([0,1]), \text{ with } h(0) = 1, h(1) = 0 ;$$

$$r_0 := \sup \{ r \mid h(r) = 1 \} \in [0,1).$$

Let $x \in D_{r_0}$ be such that $|f'(x)| = K(r_0) \left(\frac{h(r_0)}{1-r_0}\right)$; then

$$|f'(x)| = \frac{1}{1-r_0}.$$  \hspace{1cm} (4.1)

Set $\rho_0 := \frac{1-r_0}{2}$, let $x \in D(x, \rho_0) \left(\subset D_{\frac{1+r_0}{2}}\right)$; then

$$|f'(x)| \leq \|f'\|_{D_{\frac{1+r_0}{2}}} = K\left(\frac{1}{2}(1+r_0)\right) = \frac{h\left(\frac{1}{2}(1+r_0)\right)}{1-\frac{1}{2}(1+r_0)}$$

$$< \frac{1}{1-\frac{1}{2}(1+r_0)} = \frac{1}{\rho_0} \hspace{1cm} (\text{by defn. of } r_0 \text{ and } \frac{1+r_0}{2} > r_0)$$  \hspace{1cm} (4.2)

† André Bloch, not to be confused with Spencer.

Entire output produced in mental asylum after murdering his family.
\[ |f'(x) - f'(a)| \leq |f'(c)| + |f'(a)| \]
\[ < \frac{1}{\rho_0} + \frac{1}{1-r_0} = \frac{3}{2\rho_0} \]

\[ |f'(c) - f'(a)| < \frac{3|z-x|}{2\rho_0^2} \]

using Schwartz: writing \( z = 3\rho_0 + x \), \( z = \frac{z-x}{\rho_0} \), we apply it to
\[ F(z) := \frac{2\rho_0}{z} \{ f'(3\rho_0 + x) - f'(a) \} \in \mathcal{H}(\mathbb{D}), \]
which has
\[ F(0) = 0, \ |F| \leq 1 \implies |F(z)| \leq |z| \]
\[ \implies |f'(c) - f'(a)| \leq \frac{3}{2\rho_0} \cdot |z-x| \]

\[ \Rightarrow \text{ for } z \in S := \overline{D}(a, \frac{\rho_0}{3}), \]
\[ |f'(c) - f'(a)| < \frac{3\cdot\rho_3}{2\rho_0^2} = \frac{1}{2\rho_0} = |f'(a)| \]

\[ \Rightarrow f \big|_S \text{ is injective.} \]

Lemma 2

Now define \( R := \frac{\rho_0}{3}, \ \mu := \frac{1}{2\rho_0}, \ M := \frac{1}{3}, \)
\[ g(x) := f(x + \alpha) - f(a) : D_R \to \mathbb{C} \]
\[ 0 \to 0, \]
and note \( |g'(z)| = |f'(z)| = \frac{1}{2\rho_0} = \mu. \) For \( z \in D_R, \)
\[ R := [a, z + \alpha] \subset S \subset D(a, \rho_0) \]
\[ \implies |g(x)| = |\int_x^z f'(w) dw| \leq \frac{\rho_0^2}{\rho_0^2} < \frac{1}{3} = M. \]

\[ \Rightarrow g(D_R) \supset D_\sigma, \text{ where } \sigma := \frac{R^2}{6M} = \frac{(\frac{\rho_0}{3})^2}{6(\frac{1}{3})} = \frac{1}{12} \]
\[ f(S) \supset D(f(x), \frac{1}{F^2}). \]

So for \( f \in \text{Ext} \), we have what we want.

What if \( f \in \text{Hol}(D_r) \)? Setting

\[ F(x) := \frac{f(r2) - f(0)}{rf'(0)}, \text{ we have } F \in \text{Ext} \text{ hence } (\text{for some disk } S_0 \subset D) \]

\[ F(S_0) \supset \text{disk of radius } \frac{1}{F^2}, \text{ } F|_{S_0} \text{ injective} \]

\[ \Rightarrow \exists \text{ disk } S \subset D_r \text{ s.t.} \]

\[ f(S_1) = \{ \text{disk of radius } \frac{r/f'(0)}{F^2} \}, \text{ } f|_{S_1} \text{ injective} \]  

\[ \Rightarrow \exists \text{ disk } S \subset D_{\frac{1+1}{25}} \text{ on which } f \text{ is 1-1, with image } \]

\[ \text{containing a disk } \Delta \text{ of radius } \frac{r+1}{25 \cdot F^2}. \]

But then \( f|_{S_1} \text{ is 1-1, with image } \Delta \text{ of radius } \frac{r+1}{25 \cdot F^2}. \)

Looking back at the argument, one sees that the choice of \( r \) in the construction of \( S \) can be chosen continuously so that \( S_1 < S_2 < 1 \Rightarrow \)

\[ s_1 S_1 < s_2 S_2 \subset D_1, \text{ } s_1 \Delta_1 < s_2 \Delta_2 \subset f(D_0). \]
So taking $s \rightarrow 1^-$, we obtain a disk of radius $\frac{1+1}{2.72} = \frac{1}{0.72}$ in the injective image of a subset of $D$, and hence the

**Theorem (Bloch, 1925)** $f \in \mathcal{T} \Rightarrow \text{exists disk } S \subset D$

such that $f|_S$ is injective and $f(S)$ contains a disk of radius $\frac{1}{0.72}$.

The constant $\frac{1}{0.72}$ is actually a terrible lower bound; what's important is that there exists one at all, so that in the following $B$ is not zero:

**Definition 1** Given $f \in \mathcal{T}$, let

$$\beta(f) := \sup \left\{ r \left| \text{disk } S \subset D \text{ s.t. } f|_S \text{ is } 1-1 \text{ and } f(S) \supset \text{disk of radius } r \right. \right\}.$$

Then Bloch's constant is

$$B := \inf \{ \beta(f) \mid f \in \mathcal{T} \} \geq \frac{1}{0.72}.$$

**Definition 2** Given $f \in \mathcal{T}$, let

$$\lambda(f) := \sup \left\{ r \left| f(D) \supset \text{disk of radius } r \right. \right\}.$$

Then Landau's constant is

$$L := \inf \{ \lambda(f) \mid f \in \mathcal{T} \} \geq B.$$
One question that arises is: what is the meaning of this "size)? Is there actually a disk of radius $L$ in the image of $f \in \mathcal{F}_b$, or just a sequence of disks with radii approaching $L$?

**Proposition 2**

$f \in \mathcal{F}_b \Rightarrow f(D) \supseteq (\text{disk of radius } L) \supseteq (\mathcal{G}(a) (\geq L))$

**Proof:**

- Sufficient to do for $f \in \mathcal{F}_b$
- Use compactness of $f(D)$

Rest is left as an exercise.

What is known?

$0.43 < B < 0.47$, $0.5 < L < 0.55$.

In fact,

$B \leq B_0 := \frac{\Gamma(\frac{1}{\beta}) \Gamma(\frac{\nu}{\beta})}{\Gamma(\frac{\nu}{\beta})(1+\beta)^{\nu}}$, $L \leq L_0 := \frac{\Gamma(\frac{1}{2}) \Gamma(\frac{3}{2})}{\Gamma(\frac{3}{2})}$.

**Conjecture**

$B = B_0$, $L = L_0$.

There is a known function in each case having these "proposed extremal" values of $\beta$ resp. $\lambda$. 
Example 1/

Begin with the conformal isomorphism of open regions (guaranteed by AMT)

\[ f_1 : \mathbb{D} \rightarrow \mathbb{C} \setminus \mathbb{A}, \]

where \( \mathbb{A} \subset \mathbb{C} \) is the lattice given by \( 1 + \mathbb{Z}(\gamma_1, \gamma_2) \).

In fact, \( f_1 \) is a covering map, i.e. a surjection which is everywhere a local isomorphism (no "branching" \( z \mapsto z^k \)), exactly like \( \lambda : \mathbb{C} \rightarrow \mathbb{C}^{\{0,1\}} \). In both cases (\( f_1 \& \lambda \)) the domain is simply connected and so this is called the universal cover (of \( \mathbb{C} \setminus \mathbb{A} \) resp. \( \mathbb{C}^{\{0,1\}} \)). It turns out that \( f_1^{-1}(0) = \mathbb{D}_0 \), while \( \mathbb{D}_0 \) is the largest disk in the image \( \mathbb{C} \setminus \mathbb{A} \).
Example 2

Begin with the RMT-generated CE

\[ \lambda \approx \frac{\pi}{6} \]

and extend by Schwarz reflection to get

\[ f_2 : \mathbb{D} \to \mathbb{C} , \]

which is this time a branched cover. The largest disk in the image which is the 1-1 image of a disk is (again) \( \mathbb{D} \), and this time it so happens that

\[ f_2'(0) = B_0^{-1} . \]

There is work by Baerstein & Vossen checking minimality of \( \lambda (f) (= L_0) \) amongst similarly constructed functions (i.e. universal coverings of \( \mathbb{C} \) \{finite \} \{lattice \} \{close to hexagonal\}).
III. Smale's mean-value conjecture

The following problem arose in the context of applying Newton's method to determine roots of polynomials:

Given $P(z) = z + \sum_{j=2}^{n} a_j z^j$, let $z_1, \ldots, z_n$ be the zeroes of $P'(z)$, and $w_j := P(z_j)$. Put

$$\sigma(P) := \min_{1 \leq j \leq n} \frac{|w_j|}{|z_j|}.$$  

**Conjecture (Smale)** $\sigma(P) \leq \frac{n}{n+1}$, $n \geq 1$.

**Theorem (Cherzynski-Kotkowski; Smale)** $\sigma(P) \leq 4$

This will be a consequence of one of the results for schlicht/univalent functions we prove next (Köbe's theorem).