Lecture 2: Riemann Mapping Theorem

Here are three key results to bear in mind while reading these notes; \( \mathbb{U} \) denotes a region.

**Montel**: \( f_n \rightarrow f \) implies uniformly bounded; \( f_n \rightarrow f \) uniformly.

**Normal**: for any sequence \( \{f_n\} \subset \mathbb{F} \), there exists a subsequence \( \{f_{n_k}\} \)
uniformly converging on all compact subsets of \( \mathbb{U} \).

**Hermite**: Given \( \{f_n\} \subset \mathbb{F} \) uniformly converging to \( f \),
with each \( f_n \) nowhere zero on \( \mathbb{U} \), either
- \( f \) is nowhere zero on \( \mathbb{U} \)
- or \( f \) is identically zero on \( \mathbb{U} \).

**Schwarz**: Given \( f \in \mathbb{F} (\mathbb{D}_1) \), \( f(\mathbb{D}_1) \subset \mathbb{D}_2 \), \( f(0)=0 \).
Then \( |f'(0)| \leq 1 \), and if \( |f'(0)| = 1 \) then \( f \) is a rotation \( (= f(0) \circ z) \). These are the only conformal automorphisms fixing \( 0 \), and so one can say (with the above assumptions on \( f \))

\[ |f'(0)| = 1 \implies f \in \text{Aut}(\mathbb{D}_1). \]
I. The statement

Throughout this lecture, \( \Omega \) denotes a simply connected region \( \subset \subset \Omega \) which is not all of \( \Omega \).

\[ \text{RMT} \]

\( \Omega \) is biholomorphic to \( \mathbb{D} \).

\[ \text{Corollary} \]

Given \( z_0 \in \Omega \), there exists a unique function \( f \in \text{Hol}(\Omega) \) such that

- \( f(z_0) = 0 \)
- \( f'(z_0) \in \mathbb{R}_+ \)
- \( f \) is 1-to-1
- \( f \) is "onto" the unit disk \( f(\Omega) = \mathbb{D} \).

Proof of Cor. (assuming RMT):

Existence follows from RMT, and composing with one of the maps \( \phi_\alpha(z) := \frac{z - \alpha}{1 - \bar{\alpha}z} \) to translate \( f(z_0) \to 0 \), then with a rotation to make \( f(z_0) \) positive real.

\[ \text{†} \]

We'll also write \( \phi_\alpha(z) := \frac{z - \bar{\alpha}}{1 - \alpha\bar{z}} \).
For uniqueness, suppose $f \& g$ are two such functions. Then $f \circ g^{-1}$ is a holomorphic automorphism of $D_1$, hence must be of the form $e^{i\theta} \frac{z - \xi}{1 - \overline{\xi} z}$. Now,
\[
\begin{align*}
f, g : z_0 \rightarrow 0 & \Rightarrow (f \circ g^{-1})(0) = 0 \\
& \Rightarrow \xi = 0 \\
& \Rightarrow (f \circ g^{-1})(z) = e^{i\theta} z.
\end{align*}
\]
But $(f \circ g^{-1})'(z) = f'(g^{-1}(z))/g'(g^{-1}(z)) = f'(z_0)/g'(z_0) > 0$
\[
\Rightarrow e^{i\theta} = 1.
\]
So $(f \circ g^{-1})(z) = z$, i.e. $f \circ g^{-1} = id_{D_1} \Rightarrow f = g$.

Why not $C \cong D_1$? (Certainly $z \mapsto \frac{z}{1 + |z|}$ shows that $C \cong_{\text{homeo}} D_1$. ) Answer: Liouville.

The fact that Schwarz enters above is interesting, because the idea of the proof of RRT itself comes from the Schwarz Lemma: for $f: \overline{D_1} \rightarrow D_1$ holo. with $f(0) = 0$,

\begin{itemize}
  \item $f$ is bijective (hence a conformal equivalence) if $|f'(0)|$ is as large as possible.
\end{itemize}
Given $z_0 \in \mathbb{D}$, consider holomorphic functions $f: \mathbb{D} \to \mathbb{D}$ such that $f(z_0) = 0$ and $|f'(z_0)|$ is "maximal". Maybe this gives our holomorphicity!! But two questions immediately arise:

• Is the set of possible $|f'(z_0)|$ even bounded?

  
  \[
  \text{[Yes: for some } r, \quad \bar{D}(z_0,r) \subset \mathbb{D} \implies \quad |f'(z_0)| = \frac{1}{2\pi} \int_{\partial D} \frac{f(z)}{(z-z_0)^2} \, dz = \frac{2\pi}{2\pi} \frac{1}{r^2} \leq \frac{1}{r}. ]
  \]

• Is the least upper bound obtained by some function, or is the set of possible values $|f'(z_0)|$ "not closed at the top"?
II. The first proof

Write $\mathcal{H}(U, V) := \{ f \in \mathcal{H}(U) \mid f(U) \subset V \}$.

**Lemma 1:** Given $P \in U \subset C$ open, $\{ f_j \} \subset \mathcal{H}(U, D_1)$ nonempty family of functions all sending $P \to 0$.

Then there exists an $f_0 \in \mathcal{H}(U, D_1)$ which is the normal limit of $\{ f_j \} \subset \mathcal{H}$, and which satisfies

$$|f_0'(P)| \geq |f'(P)| \quad (\forall f \in \mathcal{H}).$$

**Proof:** Set $\lambda := \sup \left\{ |f'(P)| \mid f \in \mathcal{H} \right\}$, which exists by the bracketed argument on the last page.

By the definition of $\sup/\inf$, $\exists \{ f_j \} \subset \mathcal{H}$ with $|f_j'(P)| \to \lambda$. But $\{ f_j \}$ bounded by $1 \implies \exists \{ f_{j_k} \}$ converging normally, hence to $f_0 \in \mathcal{H}(U)$.

Now $|f_{j_k}'(P) - f_0'(P)| = \frac{1}{2\pi} \left| \int_{\partial(D(P, r))} \frac{f_{j_k}(z) - f_0(z)}{(z - P)^2} \, dz \right|

\leq \frac{1}{r} \left\| f_{j_k} - f_0 \right\|_{\mathcal{H}(D(P, r), C)}$

(by compactness)

$(k \to \infty) \quad \downarrow \quad 0$
Hence $|f_0'(p)| = 0$. As $f_0$ is a limit of functions in $\text{Hol}(U, D_1)$, $f_0(U) \subset \overline{D_1}$. If $\exists Q \in U$ with $|f_0(Q)| = 1$ then $\text{MMP} \Rightarrow f_0 = e^{i\theta}$ (constant of modulus 1). This contradicts $f_0(p) = 0$, and so we conclude that $f_0(U) \subset \overline{D_1}$.

Now let $\mathfrak{L}$ be as above, and set

$$\mathfrak{L} := \{ f \in \text{Hol}(\mathfrak{L}, D_1) \mid f \text{ 1-to-1,  } f(p) = 0 \}.$$  

Lemma 2: $\mathfrak{L} \neq \emptyset$.

Proof:  
- $\mathfrak{L} \subset C \setminus \{e^i\} \Rightarrow J(t) := z - e^i$ is nowhere zero  
- $\mathfrak{L}$ simply conn. $\Rightarrow \exists H \in \text{Hol}(\mathfrak{L})$ with $H^2 = J$  
- $J$ 1-to-1 $\Rightarrow H$ 1-to-1, and  
  \begin{align*}  
  \text{if } w \in \text{im}(H) \text{ then } -w \notin \text{im}(H)  
  \Rightarrow H \text{ open, with } \mathfrak{B} = D(p,r) \subset H(\mathfrak{L})  
  \text{ and } (-\mathfrak{B}) \cap H(\mathfrak{L}) = \emptyset  
  \Rightarrow \text{inversion in } (-\mathfrak{B}) \text{ maps the image of } H \text{ into } (-\mathfrak{B}) \text{ which can then be translated and dilated into } D_1.  
\end{align*}
More explicitly, \( H(z) + \beta \) has image outside the disk \( D_r \), so \( \frac{2}{r}(H(z) + \beta) \) has image outside \( \overline{D_1} \), and 
\[
 f(z) := \frac{r}{2(H(z) + \beta)} \text{ maps } \mathbb{D} \text{ into } D_1. 
\]
This is 1-to-1 (because composition of 1-to-1 with FLT) and composing \( f \) with \( \phi_{f(p)} \) (to send \( f(p) \) to 0) ensures that \( F = \phi_{f(p)} \) of sends \( P = 0 \). So \( F \in \mathbb{N} \). \( \square \)

**First Proof of RMT:** It will suffice to show

(a) \( \mathbb{N} \neq \emptyset \)

(b) If \( f_0 \in \mathbb{N} \), s.t. \( |f_0'(p)| = \sup_{h \in \mathbb{N}} |h'(p)| \)

(c) If \( g \in \mathbb{N} \) has \( |g'(p)| = \sup_{h \in \mathbb{N}} |h'(p)| \), then \( g(\overline{D}) = D_1 \).

(a)

\( \square \) done (Lemme 2)

(b)

We only need to check that the \( f_0 \) produced by Lemme 1 is 1-1.

Let \( \beta \in \mathbb{D} \) and look at
\[
 g_j(z) := f_j(z) - f_j(\beta) \in \text{slal}(D \setminus \{p\}) \]
\[
f_j \text{ 1-1 } \Rightarrow g_j \text{ is nonzero } 0. \text{ Now Hurwitz }
the normal limit of the \( \{g_j\} \) (namely, \( f_0(\alpha) - f_0(\beta) \)) is either nowhere 0 or identically 0. Suppose the latter: 
\[
0 = |f_0'(P)| = \sup \{ |h'(P)| \mid h \in \mathcal{F} \},
\]
which means that \( h'(P) = 0 \ (\forall h \in \mathcal{F} \neq \emptyset) \), contradicting that each \( h \) is 1-1.

Therefore \( f_0(\alpha) - f_0(\beta) \) must be nowhere 0 on \( \Sigma \setminus \{P\} \), meaning \( f_0(\alpha) \neq f_0(\beta) \) for \( \alpha \neq \beta \).

Since \( \beta \) was arbitrary, \( f_0 \) is 1-to-1.

\[\text{(C)}\] Take \( g \in \mathcal{F} \) with maximal \( |g'(P)| \).
Let \( Q \in \mathcal{F}_1 \) be such that \( Q \notin g(\Sigma) \).
(We are after a contradiction — i.e., to construct some \( \rho \in \mathcal{F} \) with bigger \( |\rho'(P)| \).
Set \( \phi(z) = \frac{g(z) - Q}{1 - \overline{Q} \cdot g(z)} \). This is still 1-1, with \( \phi(\Sigma) \subset \mathcal{F}_1 \), and nowhere vanishing to boot.

Together with the fact that \( \Sigma \) is simply conn., this ensures the existence of \( \Psi \in \mathcal{H}(\Sigma) \) with \( \Psi^2 = \beta \). Obviously \( \Psi \notin \mathcal{F} \) (it's still nonvanishing), so \( \rho \neq 1 \).
\[ \rho(z) = \frac{\psi(z) - \psi(P)}{1 - \psi(P) \psi(z)} , \]

which does belong to \( \mathcal{F} \).

Actually, let's "rephrase" this construction in terms of
\[ \phi_Q(z) = \frac{z - Q}{1 - Qz} , \quad \phi_Q(P) = \frac{z - \psi(P)}{1 - \psi(P) z} , \quad \text{and} \quad S(z) = z^2 : \]
\[ \rho = \phi_Q(P) \circ \psi \Rightarrow \phi_Q^{-1} \circ \rho = \psi \Rightarrow \]
\[ S \circ \phi_Q^{-1} \circ \rho = S \circ \psi = \psi^2 = \rho = \phi_Q \circ g . \]

So \( g = \phi_Q^{-1} \circ S \circ \phi_Q^{-1} \circ \rho = : \ h \circ \rho \), where
\( h : D_1 \to D_1 \) is not an automorphism (because \( S \)),
but takes \( 0 \mapsto \psi(P) \mapsto \psi(P)^2 = \phi(P) = -Q \mapsto 0 . \)

Hence \( \text{Schwarz} \Rightarrow |h'(0)| < 1 \Rightarrow |g'(P)| = |h'(0)| \cdot |\rho'(P)| < |\rho'(P)| , \)
contradicting maximality of \( |g'(P)| . \)
III. The second proof

We can construct approximate mappings of a bounded simply-connected region $\Omega$ into $D_1$ in the sense of the following

**Lemma 3 (Carathéodory):** \( \exists \{f_n\} \subset \text{Hol}(\Omega, D_1) \) s.t.
- (a) \( f_n(0) = 0 \) (for some fixed $P \in \Omega$)
- (b) \( f_n(z) \) maps $\Omega$ to a region $\Omega_n$ in 1-1 fashion, with $D_{r_n} \subseteq \Omega_n \subseteq D_1 \ (r_n \in (0,1))$.
- (c) $r_n \to 1$ as $n \to \infty$.

**Proof:** Take $\Lambda_0 = \Omega$, and define $\Lambda_1 := f_1(\Lambda_0)$, where $f_1(z) := e \cdot (z - P)$, translates & dilates $\Lambda_0$ to fit it inside $D_1$. Let $r_1 := \text{radius of the largest } D_0 \subset \Lambda_1$. Some $z_1 \in \partial D_{r_1}$ is not in $\Lambda_1$ (since $\Lambda_1^c$ is compact and $d(\Lambda_1^c, \partial D_{r_1}) = 0$).

Now inductively define, given \( \{ \Omega \rightarrow D_1 \} \),
• $f_{n+1}$ by $(\frac{z_n - f_{n,2}(z)}{1 - z_n f_{n,2}(z)})^{1/2} = \frac{z_n^{1/2} - f_{n,2}(z)}{1 - z_n^{1/2} f_{n+1}(z)}$

• $\Delta_{n+1} := f_{n+1}(\Delta_n)$

• $\rho_{n+1} :=$ radius of largest $D_r$ in $\Delta_{n+1}$

• $z_{n+1} \notin \Delta_{n+1}$ with $|z_{n+1}| = \rho_{n+1}$ (may not be unique). Notice that $f_{n+1} = (\tilde{\phi}_{e^{-n}} \circ S^{-1} \circ \tilde{\phi}_{e^n}) \circ f_n$.

Geometrically, $\tilde{\phi}_{e^{-n}}^{-1} \circ S^{-1} \circ \tilde{\phi}_{e^n}$ pushes the point $z_n$ on the boundary of $\Delta_n$ to zero, takes square root, and pushes 0 out again, to a point at distance $r_n^{1/2}$ from 0 (i.e. further out). The circle of radius $r_n$ is mapped to a lemniscate (W/interior) of smallest radius $\approx R(r_n)$, and clearly this is a lower bound for $\rho_{n+1}$.

To compute $\rho_{n+1}$, it is enough to consider $\tilde{\phi}_{e^{-n}}^{-1} \circ S^{-1} \circ \tilde{\phi}_{e^n}$, which sends

(\begin{itemize}
  \item $r_n$ \mapsto \circ \mapsto \circ \mapsto -R(r_n) \mapsto \circ \mapsto \sqrt{r_n}
\end{itemize})

(It's up to you to check that the closest point indeed has phase $\pi$.)
We have \[
\frac{r_n(1-e^{ir})}{1-r_n^2 e^{ir}} = \frac{\sqrt{r_n} - R(r_n) e^{ir}}{1 - \sqrt{r_n} R(r_n) e^{ir}}
\]
\[
\Rightarrow \sqrt{\frac{2r_n}{1+r_n^2}} = \frac{\sqrt{r_n} + R(r_n)}{1 + \sqrt{r_n} R(r_n)}
\]
\[
\Rightarrow R(r_n) = \frac{\sqrt{r_n} (r_n - 1) + \sqrt{2} r_n (1 + r_n^2)}{1 + r_n}
\] (\(\leq r_{n+1}\)).

(The point is that if the \(r_n\)-disk is contained in \(\mathcal{D}_n\), then the lemniscate I drew has to be in \(\mathcal{D}_{n+1}\).

So... locally at 0, \(R(r)\) has dominant term \((\sqrt{2} - 1) \sqrt{r} \ (\gg r)\); thus for small \(r\) the function \(R(\cdot)\) is increasing fast. Further, solving

\[r = \frac{\sqrt{r} (r-1) + \sqrt{2} r (1+r)}{1+r}
\]

leads to \(r (r^2 - 1) (6r - 1) < 0\)

meaning the graph of \(R\) looks like

so clearly \(r_n < R(r_n) \leq r_{n+m} \leq 1\) and \(r_n \to 1\).
**Second Proof of RMT:** We'll do this for bounded simply-connected \( \Omega \) — obviously enough (cf. the proof of Lemma 2):

- The \( \{f_n\} \) produced by Lemma 3 are uniformly bounded by 1; and so by Montel, some subsequence converges uniformly on (all) compact subsets. Since the limit function \( f \) is (in any such sub) a uniform limit of analytic functions, it must be analytic. Furthermore, \( f(0) = 0 \).

- The same argument as in the 1st proof shows \( f \) is 1-1.

So (as before) we must check \( f \) is onto \( D_1 \). This goes a little differently.

Take any \( w_0 \in D_1 \); \( w_0 \) lies in \( D_{1-\varepsilon} \) for some \( \varepsilon > 0 \). We may assume \( \Omega \) is bounded and obtain that the limit \( F \) of \( \{f^{-1}_n \mid n \geq N\} \) (choose \( N \) s.t. \( r_N \geq 1-\varepsilon \)) on \( D_{1-\varepsilon} \) is analytic and 1-1 (by taking a subsequence and applying Montel if Hurwitz as above). Consider the compact subset \( F(\overline{D(w_0,\varepsilon)}) = \Omega \); since \( F \) is 1-1, the interior \( F(D(w_0,\varepsilon)) \) is a neighborhood of \( F(w_0) \) (with compact closure \( \subset \Omega \)), and
so contains all \( \{ f_m(w_0) \} \) for \( m \geq M \) (\( \geq N \)). Also, the \( \{ f_n \} \) converge uniformly on the compact closure (and are continuous there), so that we may write

\[
W_0 = \lim_{n \to \infty} (f_n \circ f_n^{-1})(w_0) = \lim_{n \to \infty} \lim_{m \to \infty} (f_n \circ f_m^{-1})(w_0)
\]

\[
= \lim_{n \to \infty} f_n \left( \lim_{m \to \infty} f_m^{-1}(w_0) \right) = \lim_{n \to \infty} f_n(F(w_0)) = f(F(w_0))
\]

so that indeed \( f \) hits \( W_0 \). \( \square \)