Lecture 20: Schlicht functions I

Continuing the line of thought initiated in Lecture 19, we shall now study properties of the set

$$\mathcal{S} := \{ f \in Hol(D) \mid f \text{ 1-1, } f(0) = 0, f'(0) = 1 \}$$

of schlicht (or univalent) functions. What makes it interesting is its compactness in the topology of uniform convergence on compact sets (cf. and cf. Lecture 1 — henceforth called the normal topology), and the consequent solubility of extremal problems.

Early on in our discussion we shall connect up with a HN problem from last term: given

$$f = \sum_{n=0}^{\infty} a_n z^n \in Hol(D), \text{ and 1-to-1,}$$

we have

$$\text{Area } (f(D)) = \pi \sum_{n=2}^{\infty} n |a_n|^2 ;$$

Furthermore, if \( f \in Hol(D) \) and

\[ \begin{cases} f(0) = 0 \\ f \text{ is 1-1} \\ |f(e^{i\theta})| \geq 1 (\forall \theta) \end{cases} \]

then \( \text{Area } (f(D)) \geq \pi. \) (This isn't exactly what we'll prove below, but it's relevant.)
I. Some lemmas

Let $R \subset \mathbb{C}$ be a domain; write $D = D_1$ and "a" for "area".

Lemma 1: If $f \in \mathcal{H}(D)$ injective $\Rightarrow A(f(R)) = \int_{\partial R} |f'|^2 \, d\omega$.

**Proof:** $|f'|^2 = (\text{Re}(f'))^2 + (\text{Im}(f'))^2 = (u_x)^2 + (v_x)^2 = \det \begin{pmatrix} u_x & v_y \\ v_x & u_y \end{pmatrix}$ (C-R eqns.)

$= \det (J_f)$. Done by change-of-variable formula.

Lemma 2 (Parseval formula): If $g(\theta) = \sum_{n \in \mathbb{Z}} c_n e^{i n \theta}$ and $h(\theta) = \sum_{n \in \mathbb{Z}} d_n e^{i n \theta}$ are uniformly convergent on $R$, then

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} g(\theta) \overline{h(\theta)} \, d\theta = \sum_{n \in \mathbb{Z}} c_n \overline{d_n},$$

with sums absolutely convergent.

**Proof:** Put $S_n(\theta) := \sum_{k=-n}^{n} c_k e^{i k \theta}, \quad T_n(\theta) := \sum_{k=-n}^{n} d_k e^{i k \theta}.$

Now $\int_{-\pi}^{\pi} e^{i k \theta} e^{-i k \theta} \, d\theta = 2\pi \delta_{k0}$, and our assumption implies that $S_n \overline{T_n} \xrightarrow{n \to \infty} g \overline{h}$; so

$$\int_{-\pi}^{\pi} g(\theta) \overline{h(\theta)} \, d\theta = \lim_{n \to \infty} \int_{-\pi}^{\pi} S_n \overline{T_n} \, d\theta = \lim_{n \to \infty} \sum_{k=-n}^{n} c_k \overline{d_k}.$$

If $g = h$, then this is

$$\left(\sum_{n=-\infty}^{\infty} \right) \int_{-\pi}^{\pi} |g| \, d\theta = \lim_{n \to \infty} \sum_{k=-n}^{n} |c_k|^2,$$
\[
\sum |c_n| \leq \left( \sum |c_n|^2 \right)^{1/2} \left( \sum |d_n|^2 \right)^{1/2} < \infty.
\]

(Cauchy-Schwarz)

Remark: Lemma 1 & 2 lead (if \( D = D \)) immediately to
\( A(f(D)) = \pi \sum |a_n| a_n^1 \).

Lemma 3 (Green's formula): (a) If \( \partial \Omega \) is a \( C^1 \) Jordan curve, and \( f \in C^4(\Omega \overline{\Omega}) \), then \( \int_{\partial \Omega} f \, d\gamma = 2i \int_{\Omega} \frac{\partial f}{\partial \overline{z}} \, dx \, dy \).

(b) In particular, \( \int_{\partial \Omega^2} ^{\Omega} \overline{d} \, d\gamma = 2i \, A(\Omega) \).

Proof: \( \int_{\partial \Omega} P \, dx + Q \, dy = \int_{\Omega} (Q_x - P_y) \, dx \, dy \) becomes
\[
\left( \text{with } P = f, \quad Q = if \right)
\]
\[
\int_{\Omega^2} f \, (dx + i 
abla dy) = \int_{\Omega^2} i \left( \frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right) \, dx \, dy.
\]
II. The area formula

Given \( f \in \mathcal{A} \), from \( f(0) = 0 \) and \( f'(0) = 1 \) we have

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n.
\]

The Koebe function

\[
K(z) := \frac{z}{(1-z)^2} = z \sum_{n=0}^{\infty} \frac{(-2)^n}{n!} = \sum_{m=1}^{m=n+1} \frac{(-1)^n (-2)^n}{n!} = (-1)^n (n+1)
\]

has this form.

\[\text{Lemma 4: } K \in \mathcal{A}, \text{ and } K(D) = \mathbb{C}\setminus(-\infty, -\frac{1}{4}].\]

**Proof:** Recall \( \frac{1+z}{1-z} : D \to \mathbb{H} \), which gives

\[
\frac{1+z}{1-z} : D \to \{ \text{upper half-plane} \}, \text{ hence (by composing with}
\]

\[ (i^2 \cdot \text{RHP}) \in \mathbb{C}\setminus(-\infty, 0]\]

\[
(\frac{1+z}{1-z})^2 : D \to \mathbb{C}\setminus(-\infty, 0]. \text{ So}
\]

\[
K = \frac{1}{4} \left( (\frac{1+z}{1-z})^2 - 1 \right) : D \to \mathbb{C}\setminus(-\infty, -\frac{1}{4}],
\]

\[
= \frac{2z}{(1-z)^2}
\]

While \( K'(z) = \frac{1-z^2}{(1-z)^4} \Rightarrow K'(0) = 1 \); and \( K(0) = 0 \).
Write \( \hat{D}_r := \hat{D} \setminus D_r \), \( \hat{D} := \hat{D}_1 \). Related to \( \hat{D} \) is the set

\[
\Sigma := \{ F \in \text{Me}_2(\hat{D}) \mid F \text{ is 1-1, } F(\infty) = \infty, F(z) \neq a \text{ hole at } a \}.
\]

Notice that \( F \in \Sigma \Rightarrow F(z) = z + \sum b_n z^{-n} \quad n \geq 0 \).

**Examples**

1. \( F_0(z) = z + \frac{1}{z} \) is 1-1 on \( \hat{D} \) \( (\Rightarrow F_0 \in \Sigma_1) \), since

\[
2 + \frac{1}{z} = w \Rightarrow z^2 - w z + 1 = 0 \quad \text{has 2 solutions in } \hat{D} \quad \text{or} \quad \text{one each in } D \setminus \hat{D}.
\]

This happens for exactly those \( w \) of the form \( e^{2\pi i} + \frac{1}{e^{2\pi i}} = 2 \cos \theta \), i.e. \( w \in [-2,2] \). So \( F(\hat{D}) = C \setminus \{0,2\} \).

2. Given \( \lambda \in C \setminus D \), \( b \in C \), we have

\[
F(z) := \lambda^{-1} F_0(\lambda z) + b = z + b + \frac{\lambda^2}{z} \in \Sigma,
\]

with \( F(\hat{D}) = C \setminus \text{blobs of size at length } \lambda \) for \( |\lambda| = 1 \).

**Lemma 5:**

(i) \( F \in \delta \Rightarrow \frac{1}{F(\frac{1}{z})} \in \Sigma \)

(ii) \( F \in \Sigma \), \( \alpha \notin F(\hat{D}) \Rightarrow \frac{1}{F(\frac{1}{z}) - \alpha} \in \delta \)

(iii) \( r \in (1,\infty) \), \( F \in \Sigma \Rightarrow \gamma_r(\theta) := F(re^{i\theta}) \) is a \textbf{compact-chord} Jordan curve.
Proof (of (iii)): Let $\alpha \in \mathbb{C} \setminus F(\hat{D})$, $g(z) = \frac{1}{F(z) - \alpha}$, $e \in \mathbb{C}$.

So $\delta_r(\theta) := g\left(\frac{1}{2}e^{i\theta}\right)$ parametrizes $\partial g(D)$ and has counterclockwise orientation (because $\frac{1}{2}\int_{[0,2\pi]} \frac{d}{d\theta} g(e^{i\theta}) + \alpha = 1$ by argument principle), while $\gamma_r(\theta) = F(re^{i\theta}) = \frac{1}{g(\frac{1}{2}e^{i\theta})} + \alpha = \delta_r(\theta)^2 + \alpha$.

\[\text{Theorem 1 (Gronwall, 1914)}\]

(i) $F \in \Sigma \Rightarrow \sum_{n=1}^{\infty} n|b_n|^2 \leq 1$, with equality if \[\alpha(\partial \mathbb{C} \setminus F(\hat{D})) = 0\]

(ii) $|b_n| \leq 1$; and $|b_1| = 1 \Rightarrow b_n = 0 \forall n \geq 2$.

Proof of (i): Let $\gamma_r (r > 1)$ be as in Lemma 5 (iii), and

$\Omega_r := \mathbb{C} \setminus F(\hat{D})$. (Clearly $\gamma_r = \gamma (\partial \Omega_r)$.) Then

\[2iA(\Omega_r) = \int_{\gamma_r} F' \, d\theta\]

$= \int_0^{2\pi} F(re^{i\theta}) F'(re^{i\theta}) d(re^{i\theta})$

$= i \int_0^{2\pi} \left( \sum_{n=0}^{\infty} b_n r^n e^{in\theta} \right) \left( \sum_{n=0}^{\infty} \frac{d}{d\theta} b_n r^n e^{in\theta} \right) d\theta$

$= 2\pi i \left( r^2 - \sum_{n=1}^{\infty} n|b_n|^2 r^{-2n} \right)$

\[\Rightarrow \sum_{n=1}^{\infty} n|b_n|^2 r^{-2n} = r^2 - \frac{1}{r} A(\Omega_r)\]

\[\Rightarrow \sum_{n=1}^{\infty} n|b_n|^2 = 1 - \frac{1}{r} A(\partial \mathbb{C} \setminus F(\hat{D})) \leq 1.\]

(Use Abel's Theorem for power series)

Remark: Any $z + b_0 + \frac{e^{i\theta}}{r} \in \Sigma$, by Example 2 above.
III. The $a_2$ theorem

Given $f = z + a_2 z^2 + a_3 z^3 + \cdots \in \mathbb{H}$, we have

$$f(z) = e^z f_1(z) \text{ with } f_1 \in Hol(D), \quad f_1(0) = 1, \quad f_1 \text{ even}$$

$$\Rightarrow \exists f_2 \in Hol(D) \text{ with } f_2^2 = f_1, \quad f_2(0) = 1.$$  \[D:\text{slightly condensed}\]

Set $g(z) = e^{f_2(z^2)}$; then $g$ is odd, and

$$g(z)^2 = e^{2 f_2(z^2)} = e^{2 f_1(z^2)} = f(e^{z^2}),$$

i.e. $g(z) = \sqrt{f(e^{z^2})}$.

**Lemma 6:** $g \in \mathbb{H}$

**Proof:** \[g(z_1) = g(z_2) \Rightarrow g(z_1)^2 = g(z_2)^2 \Rightarrow f(z_1^2) = f(z_2^2) \Rightarrow z_1 = \pm z_2.\]

Suppose $z_1 \neq z_2$. Then $0 \neq z_1 = -z_2$, and

$g$ odd $\Rightarrow g(z_2) = -g(z_1)$

while\[g \neq 0, f \in \mathcal{B} \Rightarrow g(z_1) \neq 0\]

$\Rightarrow g(z_2) \neq g(z_1)$.

**Theorem 2 (Bieberbach, 1916)** $f \in \mathcal{B} \Rightarrow |a_2| \leq 2,$

with equality iff $f$ is "a rotation of the Koebe function", i.e.

$$f(z) = e^{-i \theta} K(e^{i \theta} z) \text{ for some } \theta \in \mathbb{R}.$$
Proof: With \( g \) as above, \( g(\varepsilon) = \varepsilon + A_3 \varepsilon^3 + O(\varepsilon^5) \) (for \( \varepsilon \to 0 \)).

(since \( g \in \Theta \) and \( g \) is odd). Now,

\[
(\frac{\varepsilon}{\varepsilon^2}) \Rightarrow \varepsilon^2 + 2A_3 \varepsilon^4 + O(\varepsilon^6) = \varepsilon^2 + a_2 \varepsilon^4 + O(\varepsilon^6)
\]

\[
\Rightarrow 2A_3 = a_2.
\]

Set \( G(\varepsilon) := \frac{1}{g(\frac{\varepsilon}{\varepsilon^2})} \in \Xi \) (13.1.1)

\[
= \frac{1}{\frac{1}{\varepsilon} + \frac{A_3}{2} \varepsilon + \ldots} = \frac{\varepsilon}{1 + A_3 \varepsilon^2 + \Theta(\varepsilon^4)}
\]

\[
= \varepsilon - A_3 \varepsilon^{-1} + \Theta(\varepsilon^{-2}) \quad \text{(for \( \varepsilon \to 0 \))}
\]

By Theorem 1, \( |A_3| \leq 1 \) hence, \( |a_2| \leq 2 \).

Next, if \( |a_2| = 2 \), then \( |A_3| = 1 \) \( \Rightarrow \)

\[
G(\varepsilon) = \varepsilon - \frac{A_3}{\varepsilon} = \varepsilon - \frac{\varepsilon^2}{\varepsilon} \Rightarrow g(\varepsilon) = \frac{1}{G(\frac{\varepsilon}{\varepsilon^2})} = \frac{1}{\frac{1}{\varepsilon} - \varepsilon^2} = \frac{\varepsilon}{1 - \varepsilon^2 \varepsilon^2}
\]

\[
\Rightarrow f(\varepsilon^2) = (g(\varepsilon))^2 = \frac{\varepsilon^2}{(1 - \varepsilon^4 \varepsilon^2)^2} \Rightarrow f(\varepsilon) = \frac{\varepsilon^2}{(1 - \varepsilon^4 \varepsilon^2)^2} \text{ as claimed.}
\]
IV. The $\frac{1}{4}$ theorem

No, it isn't true that $0 < \frac{1}{4} < 1$. (That's next lecture.)

Theorem 3 (Köbe, 1907)

(i) $f \in S \Rightarrow f(0) > D_{\frac{1}{4}}$

(ii) If (for $f \in S$) $f(0) \not= D_{\frac{1}{4}}$, then

$$f(z) = e^{-i\alpha} K(e^{i\alpha} z)$$

for some $\alpha \in \mathbb{R}$.

Proof: Let $b \in \mathbb{C} \setminus f(0)$. Since $b \not= 0$,

$$g(z) = \frac{f(z)}{(-b/f(z))} \in \mathbb{H}(D)$$

with $g(0) = 0$ and $g'(0) = 1$.

Since $g = \text{FLT}_f$, $g$ is $1$-1; so $g \in S$.

Write $g = \frac{z + a_2 z^2 + O(z^3)}{1 - \frac{a_1}{b} z + O(z^2)}$

$$= (z + a_2 z^2 + O(z^3))(1 + \frac{a_1}{b} z + O(z^2))$$

$$= z + (a_2 + \frac{a_1}{b}) z^2 + O(z^3).$$

Theorem 2(i) $\Rightarrow |a_2 + \frac{a_1}{b}| \leq 2$ and $|a_2| \leq 2$

$\Rightarrow \frac{1}{|b|} = |(\frac{1}{b} + a_2) - a_2| \leq |\frac{1}{b} + a_2| + |a_2| \leq 4$

$\Rightarrow f(0) > D_{\frac{1}{4}}$ (as $b$ was an arbitrary point not in $f(S)$).

Now, for the extremal case: if $f$ is not a rotation of $K$,

then Theorem 2(ii) $\Rightarrow$
\[ |a_0| = 2 - \varepsilon \quad \text{for some } \varepsilon \in (0, 2] \]
\[ \frac{1}{|b|} \leq 2 + (2 - \varepsilon) = 4 - \varepsilon \]
\[ |b| \geq \frac{1}{4 - \varepsilon} > \frac{1}{4} \quad \Rightarrow \quad f(0) > \overline{D}_{4^4} . \]

There is a nice application to Smale's conjecture (end of lecture 19): Let \( p(x) = 2 + \sum_{k=2}^{n} a_k x^k \) (note: \( p(0) = 0 \), \( p'(0) = 1 \)), \( \{x_1, \ldots, x_n\} = 0's \) of \( p' \), \( \{w_1, \ldots, w_n\} = \) their image under \( p' \), \( \sigma(P) := \min_{1 \leq j \leq n} \frac{|w_j|}{|x_j|} \). Conjecture is that \( \sigma(P) \leq \frac{n}{n+1} \) (known for \( n \leq 3 \)).

**Theorem 4 (Czerwiński-Kawosi/Smale)** \( \sigma(P) \leq 4 \).

**Proof:** [Note: read appendix on covering maps first.]

First, \( \mathcal{S}_2 = C \setminus \{w_1, \ldots, w_n\} \)
\[ \mathcal{P} \leftarrow \mathcal{S}_1 = p^{-1}(\mathcal{S}_2) \subset C \setminus \{x_2, \ldots, x_n\} \]

is an analytic covering map, since zeroes of \( p' \) are omitted. Wolog assume \( M := \{w_1 < |w_2| < \cdots < |w_n| \} \).

Applying the lifting theorem to
\[ f \rightleftharpoons \mathcal{S}_1 \]
\[ \mathcal{P} \leftarrow \mathcal{S}_2 \]
\[ (\text{inclusion}) \]

yields \( f \circ i = \mathcal{P} \circ f \). Note that \( i \) injective \( \Rightarrow \mathcal{P} \) injective.
Moreover, we can choose $f$ so that $0$ is sent to any point in $P^{-1}(0)$; so assume $f(0)=0$.

Set $f_i(t) := \frac{f(tM)}{M} \in \mathcal{ML}(D)$. We have

$$1 = i'(0) = P'(f(0)) \cdot f'(0) = P'(0) \cdot f'(0) = f'(0)$$

$\Rightarrow f_i'(0) = f'(0) = 1 \quad \Rightarrow f_i(0) = 0$

$\Rightarrow f_i \in \mathcal{S}$.

Since $\exists j \notin \mathcal{S}$, $f(D_M)$ contains no $\pm j$'s.

$\Rightarrow f_i(D) \left( = \frac{1}{M} f(D_M) \right)$ contains no $\frac{\pm i}{M}$.

By Köbe-$\frac{1}{q}$, $f_i(D) \supset D_{1+q}$, and so each $\left| \frac{\pm i}{M} \right| \geq \frac{1}{q}$.

In particular, $1+1 \geq \frac{M}{q} = \frac{|w|}{q} \Rightarrow \min \frac{|w_j|}{|e_j|} \leq \frac{|w|}{|e|} \leq 4$. \hfill $\Box$
V. A first look at distortion

In the next lecture we will prove Kôbe’s Distortion Theorem (1907), part of which says:

Given \( f \in \mathbb{D} \) and \( z \in \mathbb{D} \),

\[
|f(z)| \leq \frac{|z|}{(1-|z|)^2}
\]

Let \( R \in \mathbb{R}_+ \) and

\[
\mathcal{S}_R := \left\{ f \in \text{Hol}(D_R) \mid f(1) = 1, f(0) = 0, f'(0) = 1 \right\}
\]

Corollary \( \mathcal{S}_R \) is compact in the normal topology; that is, any sequence has a subsequence normably converging to an element of \( \mathcal{S}_R \) itself. (This is stronger than normality.)

Proof: Compactness of \( \mathcal{S}_R \) will follow from that for \( \mathbb{D} \) via the homeomorphism

\[
\mathcal{S}_R \to \mathbb{D},
\]

\[
f(z) \to \frac{f(Rz)}{R}.
\]

According to Montel’s theorem, since \( \mathbb{D} \) is locally
bounded (i.e. on each $\overline{D_r}$, $r < 1$), we have that any sequence $\{f_n\} \subset B$ has a normally convergent subsequence $f_{n_k} \to f$. Clearly $f(0) = 0$, $f'(0) = 1$ ($\Rightarrow f$ non-constant).

To see $f$'s injectivity, we must show that $f - \beta$ has at most one zero for any $\beta \in \mathbb{C}$, which follows from the observation ($\forall r < 1$, except for values of $1$)

where $f(\gamma) = \beta$:

\[
(0 \leq) \frac{1}{2\pi i} \int_{\partial D_r} \frac{f'}{f - \beta} \, dz = \frac{1}{2\pi i} \oint_{\partial D_r} \frac{f_{n_k}'}{f_{n_k} - \beta} \, dz \leq 1.
\]

So $f \in A$. \qed
Appendix: Covering Spaces

Let $\Omega_1 \xrightarrow{F} \Omega_2$ be a continuous mapping of topological spaces. $F$ is called a covering map if it is surjective and a local homeomorphism. When $F$ is an analytic map of regions (or Riemann surfaces), this will mean that a sufficiently small disk $D$ about each point in $\Omega_2$ has preimage equal to a (nonempty) disjoint union of disk-like blobs in $\Omega_1$, each of which $F$ maps 1-1 conformally onto $D$. Equivalent conditions are that the derivative $F'$ be everywhere nonzero ("F étale"), or that there be no ramification points (when $z \mapsto e^{kz^k}$ locally); this is enough because of the inverse mapping theorem.

**Lifting Theorem**

Given $\Omega_0 = \text{simply connected region}$ and $g : \Omega_0 \rightarrow \Omega_2$ holomorphic,

then exists a (holomorphic) "lifting" map $G : \Omega_0 \rightarrow \Omega_1$ such that $F \circ G = g$.

Moreover, given $t_0 \in \Omega_0$ and any $w_0 \in F^{-1}(g(t_0))$, we may arrange that $f(t_0) = w_0$. 

![Diagram showing the lifting theorem](image)
Sketch: Taking a sufficiently small ball $B$ about $x_0 \in \mathbb{R}^n$ and composing $g$ with a local branch of $F^{-1}$ on $g(B)$, glue a germ $G_0$ at $x_0$. Any path on $\mathbb{R}_2$ lifts uniquely, after fixing the initial point's lift, to a path on $\mathbb{R}_1$ — which may not be closed even if the one on $\mathbb{R}_2$ is.

In particular, taking a path in $\mathbb{R}_0$ from $x_0$, we can cover it with balls sufficiently small that their $g$-images have homeomorphic preimages under $F$ covering the lifted path. In this way one gets an analytic continuation of $G_0$ along all paths in $\mathbb{R}_0$, and since $\mathbb{R}_0$ is simply connected, we are done by the monodromy theorem.