Lecture 22: Green’s Functions on Riemann surfaces

I. Riemann surfaces and universal covers

Definition (i) Let \( M \) be a connected Hausdorff space, \( \mathcal{U} = \{ U_\alpha \mid U_\alpha \subset M \} \) an open cover of \( M \),
\[ z_\alpha : U_\alpha \rightarrow \mathbb{C} \]
continuous 1-1 maps such that the homeomorphisms \( z_\beta \circ z_\alpha^{-1} : z_\alpha(U_\alpha \cap U_\beta) \rightarrow z_\beta(U_\alpha \cap U_\beta) \) are (bi)holomorphic. Then \( \mathcal{A} = \{ (z_\alpha, U_\alpha) \} \) is a conformal atlas, 
\((M, \mathcal{A})\) a Riemann surface (usually denoted “\( M \)”).

(ii) A function \( f \) on \( M \) is holomorphic \[ \iff \{ f \circ z_\alpha^{-1} \} \text{ are holomorphic.} \]

(iii) A map \( F : M \rightarrow M' \) of RS’s is holomorphic \[ \iff \{ z_\beta' \circ F \circ z_\alpha^{-1} \} \text{ are holomorphic.} \]

This turns out to be equivalent to the “multivalued analytic function element” approach, simply because one can always produce a centered pair of holomorphic maps \( M \rightarrow \mathbb{P}' \) such that \( M \rightarrow \mathbb{P}' \times \mathbb{P}' \) is an immersion (with finitely many transverse crossings in the image).
Every RS $M$ has a universal cover
$$\tilde{M} \rightarrow M,$$
which is a simply connected covering space. One first constructs this topologically, then lays on the holomorphic structure. One may view $M \cong p\tilde{M}$ as the quotient of $\tilde{M}$ by a group of “deck transformations” (parametrizing branches of $\tilde{M}$ over $M$), acting properly discontinuously (so that the quotient is Hausdorff). This makes sense because the preimage of a sufficiently small neighborhood of $M$ is a tower of neighborhoods on $\tilde{M}$, and $\Gamma$ acts by holomorphic automorphisms of $\tilde{M}$.

Suppose now $\tilde{M} = \mathbb{C}$. Recall that $\text{Aut}(\mathbb{C})$ consists of transformations of the form $z \mapsto az + b$. This has 3 types of “properly discontinuous subgroup”:

\[ \text{that is, every } x \in \tilde{M} \text{ has a nhbd. } U \text{ s.t. } gU \cap U = \emptyset \Rightarrow g = \text{id}. \]

To ensure the quotient is Hausdorff, we actually need a little more: for $x, x' \in \tilde{M}$ not in the same $\Gamma$-orbit, $\exists$ nhbd. $U, U'$ s.t.

\[ gU \cap U' = \emptyset \quad (\forall g \in \Gamma). \]

\[ \text{recall this means that } \varphi \text{ is surjective (onto) \& a local homeomorphism — no ramification allowed!!} \]
• trivial case \( \{e\} \Rightarrow M = \mathbb{C} \)
• \( \mathbb{Z}\langle \alpha \rangle \ (\alpha \in \mathbb{C}^* \text{ fixed}) \Rightarrow M = \mathbb{H} / \mathbb{Z} \cong \mathbb{C} \stackrel{E}{\rightarrow} \mathbb{C}^* \)
• \( \mathbb{Z}\langle \alpha, \beta \rangle \ (\alpha, \beta \in \mathbb{C}^*) \Rightarrow M = \mathbb{A} \subset \mathbb{C} \), ell. curve \( E_{\Delta} \)

Next suppose \( \widetilde{M} = \hat{\mathbb{C}} \) (i.e. \( \mathbb{P}^1 \)). We had \( \text{Aut} \hat{\mathbb{C}} \cong \text{PSL}_2(\mathbb{C}) \), but these are no "properly discontinuous subgroups" as all of these automorphisms have fixed points (blame \( \hat{\mathbb{C}} ^{\text{scheme}} \) 's compactness). So the only option here is \( M = \hat{\mathbb{C}} \).

The **Uniformization Theorem**, which we'll prove in the next lecture, essentially says that \( \hat{\mathbb{C}}, \mathbb{C}, \mathbb{C}^* \), and \( E_\Delta \) is the complete list of RS's whose universal cover is not \( \mathbb{D} \) (equality \( T_\infty \)). You may be familiar with the topology result that every compact orientable real 2-manifold, hence every compact RS, is a "sphere with handles attached":

\[
\begin{array}{c}
\hat{\mathbb{C}} \\
genus g = 0 \\
\end{array}
\begin{array}{c}
E \\
genus g = 1 \\
\end{array}
\begin{array}{c}
\text{etc.} \\
genus g = 2 \\
\end{array}
\]

Uniformization asserts that any compact RS of genus \( g \geq 2 \)
(or for that matter $E^{\nu}$ or $\hat{E}^{\nu}$) is $\rho \mathcal{D}$ (or $\rho \mathcal{L}$) for some $\Gamma \subset \text{Aut}(\mathcal{D}) \equiv \text{Aut}(\hat{\mathcal{D}}) \equiv \text{PSL}_2(\mathbb{R})$. The Poincaré metric on $\mathcal{D}$ induces through this a metric of constant negative curvature on all but the 4 "exceptional" types of $RS$ above!

Though others, including Köbe and Poincaré, made the proof more rigorous and natural, the idea of uniformization (and its proof) were due to Klein in 1882. The way he thought of it was that every $RS$ $M$ could be parametrized by a single complex variable defined on a subset of $\hat{\mathbb{C}}$. That means you can write $(sy)$ all meromorphic functions on $M$ in terms of this variable. We've already seen how powerful this is in the case of elliptic functions (defining functions on $E$) and modular functions (defining functions on modular curves like $\Gamma(N) \backslash \mathbb{H}$). The rest of this lecture, then, is in preparation for proving this important result.

In the sequel, we assume all $RS$s open.
II. Green's functions

Let $M$ be a RS, and let $V \subseteq \mathcal{H}(M)$ be a subset ("family"). (This time we will allow our subharmonic functions to tend to $-\infty$ at isolated points; but we will otherwise still take them to be continuous.) Note that the notion of subharmonicity is preserved under "precomposition" with holomorphic functions (viz., $u \circ f$); otherwise the notion wouldn't make sense on RSs.

**Definition** $V$ is **Perarn** $\iff$

\[\begin{align*}
1) & \quad \forall v_1, v_2 \in V \Rightarrow \max(v_1, v_2) \in V \\
2) & \quad \forall v \in V, A \subseteq M \text{ Jordan region implies } v_A \in V.
\end{align*}\]

**Proposition 1** $V$ **Perarn** $\Rightarrow u := \sup_{v \in V} v$ is either identically $+\infty$ or belongs to $\mathcal{H}(M)$.

**Proof**: We did this for plane regions; generalizes easy to RS. $\square$

**Definition** Let $p_0 \in M$, $z$ a local coordinate about $p_0$, 

$V_{p_0} := \text{family of } v \in \mathcal{H}(M[p_0]) \text{ s.t. } \{v \in 0 \text{ outside a compact set, and } \lim_{p \to p_0} \{vp) + \log|2(p)|\} < \infty.}$

† Last of Step 2 in the proof of Dirichlet (Lecture 7)
If $\sup v$ is finite, we say $M$ has a Green's function at $p_0$, denoted $g(p, p_0)$.

Remark: 
(i) If a GF $\exists$ at $p_0$, the range of $\varphi(p)$ contains some $\text{Dr}_0$. Set $v_0(p) = \log \frac{r_0}{\varphi(p)}$ where $\varphi(p) \leq r_0$ and $v_0(p) = 0$ elsewhere $\Rightarrow v_0 \in V_{p_0}$

$\Rightarrow g(p, p_0) \geq v_0(p) \Rightarrow \lim_{p \to p_0} g(p, p_0) = +\infty$

($\Rightarrow$ non-constant)

(ii) $M$ compact $\Rightarrow \not\exists$ Green's func. at any $p_0$.

(Reason: $g(p, p_0)$ would have a minimum, hence be constant)

(iii) $g(p, p_0) > 0$ (since $0 \in V_{p_0}$, $\geq 0$ is clear; and $M$ open $\Rightarrow g$ can't attain its minimum).

Now let $K \subset M$ be compact w/nonempty interior and $M \setminus K$ connected,

$\varphi_{K} < \varphi(M \setminus K)$ the family such that:

\begin{align*}
\{ v \in V_K : & \text{ is } \leq 1 \text{ on } M \setminus K \\
& v \in V_K \Rightarrow \lim_{p_0 \to \infty} v(p_0) = 0 \text{ when } p_0 \to \infty
\end{align*}

Clearly this has $0 \leq u_K \leq 1$. Looking at the particular elements of $V_{K}$ defined by

$v(p) = \begin{cases} 
\log_2 \left( \frac{2d}{\varphi(p) - 80} \right) & \text{if } p \in K \\
0 & \text{otherwise}
\end{cases}$
we see $v_K \neq 0$, hence $v_K > 0$ (by the minimum principle for harmonic).
Similarly if $u_K \neq 0$ then $u_K < 1$.

**Definition**  
(a) If $u_K \neq 1$ then call it the harmonic measure of $K$ (i.e. this exists).
   
(b) Say the maximum principle is valid on $M \setminus K$ if for any bounded-above $u \in K(M \setminus K),$
   
\[
\lim_{p \to K} u(p) = 0 \implies u \leq 0 \text{ on } M \setminus K. 
\]

**Proposition 2**  
The following are equivalent:

(i) Green's functions exist ($\forall p_0 \in M$)

(ii) Harmonic measures exist ($\forall K \subset M$)

(iii) The maximum principle is NOT valid ($\forall M \setminus K$).

Moreover, $\exists$ of GF [resp. HM] for any $p_0$ [resp. any $K$]
ensures their existence for all $p_0$ [resp. $K$].

**Proof:**

**STEP 1**  
\[(ii) p_0 \implies (iii)_K \forall p_0 \in K :\]

$-g(p,p_0)$ has a maximum $\mu$ on $K$, and is bounded above by 0 on $M \setminus K$. The maximum principle for $M \setminus K$ implies

\[-g \leq \mu \implies \mu = \text{ maximum value for } -g \text{ on } M \setminus p_0,
\]

assumed at some point of $K \setminus p_0 \implies g$ constant.
STEP 2 \( (ii) \Rightarrow (i)_{p_0} \) if \( p_0 \in \text{int}(K) \):

If \( u_{K_1} \not \equiv 0 \), then \( \sup_{v \in V_{K_1}} \neq 1 \Rightarrow u_{K_1} \not \equiv 0 \). So assume now that \( u_{K_1} \) exists, we have \( u_{K_2} \) too.

Now given \( v \in V_{p_0} \), we consider \( u^+ := \max \{v, 0\} \in V_{p_0} \).

Notice that \( u^+(p) \leq (\max u^+)(\max u_{K_1}) \) holds on \( \partial K_1 \), and also near the "ideal boundary" \( (p \to 0) \) [since \( u^+ \in V_{p_0} \) vanishes outside a compact set]. So then this \( \leq \) must hold outside \( K_1 \) by the maximum principle for \( M \setminus K_1 \), and we have in particular

\[ (*) \quad \max_{\partial K_2} u^+ \leq (\max_{\partial K_1} u^+)(\max_{\partial K_2} u_{K_1}). \]

Next look at \( u^+(p) + (1 + \epsilon) \log |v(p)| \) on \( K_2 \), which \( \to -\infty \) as \( p \to p_0 \) (since \( \epsilon > 0 \)). It follows that

\[ (**) \quad \max_{\partial K_2} u^+ + (1 + \epsilon) \log \rho \leq \max_{\partial K_2} u^+ + (1 + \epsilon) \log \rho_{K_1}. \]

Now \((**)+(*)\) \( \Rightarrow \)

\[ \max_{\partial K_1} u^+ \leq \max_{\partial K_1} u^+ + (1 + \epsilon) \log \left( \frac{\rho}{\rho_{K_1}} \right) \]

\[ \Rightarrow \max_{\partial K_1} u^+ \leq (1 + \epsilon) \log \left( \frac{\rho}{\rho_{K_1}} \right) \]

\[ \Rightarrow \max_{\partial K_1} u^+ \leq \frac{\log \left( \frac{\rho}{\rho_{K_1}} \right)}{1 - \max_{\partial K_2} u_{K_1}} < 1 \]

\[ \Rightarrow u^+ \], hence \( u_{K_1} \), is uniformly bounded above on \( \partial K_1 \)

\[ \Rightarrow g(p, p_0) \text{ exists.} \]
STEP 3 \( (iii)_{k} \Rightarrow (\ast)_{k}, \forall k, k' \): (This will finish the proof.)

We show \( (ii)_{k} \Rightarrow (\ast)_{k} \); so assume \( u_{k} \not\equiv 0 \), i.e.
\[
\sup_{v \in V_{k}} v = 1.
\]

For every \( k' \subset k \). Let \( v \in V_{k} \), and \( u \in H(M\backslash K) \) with \( u \leq 1 \) and \( \lim_{\rho \to k} u(\rho) \leq 0 \). Then \( v + u \leq 1 \) as \( p \to \infty \),
\[
p \to k \Rightarrow v + u \leq 1 \quad \text{on} \quad M\backslash K \quad \text{(max principle).}
\]
So taking \( v \) arbitrarily close to \( 1 \), we find \( u \leq 0 \). Hence the maximum principle is valid on \( M\backslash K \).

If \( k' \not\subset k \), choose \( k'' \) s.t. \( k \cup k' \subset \text{int}(k'') \). The assumed nonexistence of \( u_{k} \Rightarrow \text{mp} \) holds for \( M\backslash K'' \) by the last paragraph. Let \( u \in H(M\backslash K) \) with \( u \leq 1 \), \( \lim_{\rho \to k} u(\rho) \leq 0 \).
We want to show \( u \leq 0 \) (\( \Rightarrow \text{mp} \) for \( M\backslash K'') \). First, by mp for \( M\backslash K'' \),
\[
u_{|_{M\backslash K''}} \leq \max_{\partial K''} u.
\]

Now suppose \( \max u > 0 \). By the usual maximum principle in \( k'' \backslash k \), we would have
\[
u_{|_{k'' \backslash k}} \leq \max_{\partial k''} u.
\]

By the 2 displayed estimates, \( u \) attains its maximum at a point of \( \partial k'' \), which is \( \text{int} \) to \( M\backslash K \) a constant. This
is a contradiction, since it is \( > 0 \) somewhere on \( \partial k'' \) and
\( \leq 0 \) elsewhere. Hence, \( \max u \leq 0 \) and by the usual maximum principle in \( k'' \backslash k \), \( u_{|_{k'' \backslash k}} \leq 0 \). Also \( u_{|_{M\backslash K''}} \leq 0 \),
and so \( u \leq 0 \).  

\[ \square \]
Corollary 2. If it exists, \( g(p, p_0) \) satisfies the 3 properties:

(I) \( g(p, p_0) > 0 \)

(II) \( \inf g(p, p_0) = 0 \)

(III) \( g(p, p_0) + \log |\varepsilon(p)| \) has \( \epsilon \) harmonic extension to a nbhd. of \( p_0 \).

Proof: We already know (I). For (III), let \( m(r) := \max g(p, p_0) \).

\[ \begin{align*}
& (\text{HK}) \Rightarrow m(r) + \log(r) \text{ is an increasing function of } r \\
& \Rightarrow g(p, p_0) + \log |\varepsilon(p)| \text{ is bounded above near } p_0.
\end{align*} \]

Also, \( \nu(p) = \begin{cases} -\log |\varepsilon(p)| + \log r_0, & |\varepsilon(p)| < r_0 \in V_{p_0} \\ 0, & \text{otherwise} \end{cases} \)

\[ \Rightarrow g(p, p_0) \geq -\log |\varepsilon(p)| + \log r_0 \]

\[ \Rightarrow g(p, p_0) + \log |\varepsilon(p)| \text{ bounded below near } p_0. \]

Since isolated singularities of a bounded harmonic form an removable, done.

For (II), set \( c = \inf g(p, p_0) \). By (III), \( g(p, p_0) + \log |\varepsilon(p)| \) has a finite limit as \( p \to p_0 \). So (by \

\[ (1-e)\nu(p) \leq g(p, p_0) - c \quad (\forall v \in V_{p_0}, e > 0) \]

growth bad. as \( p \to p_0 \) by \( -(1-e)\log |\varepsilon(p)| \); \n
also, \( 0 \) outside compact set \n
\[ (1-e)g(p, p_0) \leq g(p, p_0) - c \quad (\forall e > 0) \]

(Def. of \( g \))

\[ \Rightarrow c \leq 0 \quad \Rightarrow \quad c = 0. \]

(Def of \( c \); fact \( \inf g(p, p_0) > 0 \)).
**Corollary 2** If $\Omega$ bounded nonconstant $h \in H(\Omega)$ then

Every compact $K \subset M$, w/muncountably dense, the maximum principle on $M$, fails, and $M$ has a Green's function w/singularity at any point $p \in M$.

**Proof:** $h$ has a maximum on $K$, attained (necessarily) on $K$. Were $MP$ for $M/K$ valid, it would follow that this was a maximum on all of $M$, rendering $h$ constant.

Now recall the Dirichlet principle. The proof generalizes immediately to Riemann surfaces in the following sense:

**Proposition 3** Let $M \subset \tilde{M}$ be RSs. If $\partial M \subset \tilde{M}$ is a finite union of closed analytic arcs, then a bounded $C^0$ functor $f : \partial M \to \mathbb{R}$, $\exists$ $h : M \to \mathbb{R}$ s.t.

- $|h| \leq \sup |f|$
- $h|_{\partial M} \subset \Omega(M)$
- $h|_{\partial M} = f$.

**Corollary 3** Under the same hypothesis,

(i) $M$ has Green's function $g(p,p_0)$

(ii) $\lim_{p \to \partial M} g(p,p_0) = 0$. 
Proof: (i) is an immediate consequence of Prop. 3 + Cor. 2.

For (ii), let $C$ be a small circle about $p_0$. Prop. 3 $\Rightarrow$

If solution $h$ to the Dirichlet problem outside $C$ w/boundary data $h|_C = g$, $h|_{\partial C} = 0$. By the maximum principle for $h$, $h = \sup_{\Omega} h$ for all $v \in V_{p_0}$. $\Rightarrow \lim_{r \to 0} g(\rho, p_0) = 0$

$\Rightarrow \lim_{r \to 0} g(\rho, p_0) \geq 0$. 

$0$-function $CV_{p_0}$