Lecture 23: The Uniformization Theorem

Let $M$ be a Riemann surface. Recall that Green’s functions $g(p, p_0)$ exist [resp. don’t exist] for all points $p, p_0 \in M$ (if harmonic measures exist [resp. don’t exist] for all compact sets $K \subset M$). We will say “$M$ has Green’s functions.”

**Definition** If $M$ is noncompact with Green’s functions, we call $M$ hyperbolic. If $M$ is noncompact w/o Green’s functions, we call $M$ parabolic.

We will prove our main result in the following form:

**Uniformization Theorem** Assume that $M$ is simply connected. Then $M$ is conformally isomorphic to $D$ (if hyperbolic), the plane $\mathbb{C}$ (if parabolic), or the Riemann sphere $\hat{\mathbb{C}}$ (if compact).

The proof is broken into 3 sections, one for each of these cases.
I. Hyperbolic case

Let $M' = M \setminus \{p_0\}$, and choose a simply connected nbhd. $N_p \subset M'$ about each $p \in M'$, so that $g := g(-, p_0)$ has a well-defined harmonic conjugate there; call this $h_p$. Set

$$f_p := e^{-(g + ih_p)} \in \mathcal{F}(N_p);$$

this is unique up to multiplication by an $e^{i\theta}$.

Now let $N_0 \subset M$ be a simply connected nbhd. about $p_0$, with local coordinate $z$. We know that $g(z, p_0) + \log(2\pi) \log(\sin |z|)$ belongs to $\mathcal{H}(N_0)$, so it has a well-defined harmonic conjugate $h_0$ there; and we can put

$$f_0 := e^{-(g + \log(2\pi) + ih_0)} \in \mathcal{F}(N_0).$$

This vanishes to first order at $p_0$.

Given any curve $\gamma$ from $p_0$ to $p$, by adjoining $e^{i\theta}$'s the $f_p$'s are analytic continuation of $f_0$. Since $M$ is simply connected, the monodromy theorem tells us that they patch together to give a global analytic
function $f : M \to C$.

Next, $|f(p_1)| = e^{-g(p,p_0)} < 1$ since $g(p,p_0) > 0$.

It suffices to prove if is 1-to-1, for then one can get an isomorphism of $M$ onto a bounded plane region, whereupon the Riemann mapping theorem produces an isomorphism $M \cong D$. Write $f(p,p_0)$ for $f$.

Consider (for some $p_1 \in M \setminus \{p_0\}$)

$$F(p) := \frac{f(p,p_0) - f(p_1,p_0)}{1 - f(p_1,p_0) f(p,p_0)} \in \text{Hol}(M)$$

$$= \phi f(p_0,p_0)(f(p,p_0)),$$

with $F(p_1) = 0$ and $|F| < 1$ (why?). If $e_1$ is a local coordinate at $p_1$, and $\psi \in \mathcal{V} p_1$ (in the notation of Lecture 22 III), then $\varepsilon > 0 \Rightarrow$

$$\lim_{p \to p_1} \left[ \psi(p) + (1+\varepsilon) \log |F(p)| \right] = -\infty.$$

By the maximum principle for $\mathcal{X}(M)$, together with $|F| < 1$ and $\lim_{p \to p_0} \psi = 0$, $\psi + (1+\varepsilon) \log |F(p)| \leq 0$ on $M$.

Taking $\varepsilon \to 0$ and $\psi \to g$, we have

$$g(p,p_1) + \log |F(p)| \leq 0$$
hence $|F(p)| \leq |f(p, p_1)|$. For $p = p_0$, this gives

$$|f(p_1, p_0)| = |F(p_0)| \leq |f(p_0, p_1)|.$$

But the argument is symmetric in $p, p_0$, so we get the reverse inequality hence

$$|f(p_1, p_0)| = |f(p_0, p_1)|.$$

Conclude that (f) is an equality for $p = p_0$, i.e.
$g(p, p_1) + \log|F(p_0)| = 0$. But then LHS ($t$) attains its maximum at $p = p_0$. Since LHS ($t$) $\in \mathcal{H}(M)$, LHS ($t$) $\equiv 0$ by the maximum principle.

$$\implies |F(p)| = |f(p, p_1)|$$

$$\implies F(p) = e^{i\Theta} f(p, p_1) \text{ w/constant } \Theta \in \mathbb{R}$$

$$\implies F(p) \text{ has only one at } p = p_1.$$
II. Parabolic case

Henceforth we assume $M$ is simply connected, without Green's functions.

**Definition** A divergent curve on $M$ is a piecewise analytic simple arc $g : [0, \infty) \to M$ s.t. $g^{-1}(K)$ is compact for every compact $K \subset M$.

Assume $M$ admits one, and set $M_t := M \setminus g([t, \infty))$.

**Lemma A:** For $t > 0$:
- $M_t$ is simply connected.
- For $p_0 \in M_0$, $M_t$ admits a GF w/singularity at $p_0$.
- $\lim_{p \to M_t} g(p, p_0) = 0$.

**Proof:** The latter two bullets follow at once from Cor. 3 of Lecture 22. To check the first, let $Y \subset M_{t_0}$ be a loop based at $p_0$, and set $A(Y) := \{ t \in [t_0, \infty) \mid \sigma \sim [p_0] \text{ in } M_{t_0} \}$. Now $Y \sim \{p_0\}$ on $M$, and the homotopy has compact image, so $A(Y)$ is nonempty and open in $[t_0, \infty)$. If $t, t' \in A(Y)$, then $[t, t'] \subset A(Y)$; set $t_2 := \inf\{ t \mid t \in A(Y) \}$, so that $Y \sim \{p_0\}$ in $M_{t_2 + e}$.

Consider a disk $\Delta \subset M_{t_2}$ about $g(t_2)$ also containing $g(t_2 + e)$. There is a $C^\infty$ automorphism of the
Closed disk which is the identity on the boundary and maps \( \gamma([t, t+\omega]) \cap \Delta \) onto \( p([t, t+\omega]) \cap \Delta \).

Since the homotopy doesn't meet the former, its composition with this automorphism (on \( \Delta \)) won't meet the latter. Thus \( \gamma \sim \{p_0\} \) in \( M_{t_0} \iff A(y) \approx_t t \iff A(y) \) is closed \( \Rightarrow A(y) = [t_0, \infty) \iff \gamma \sim \{p_0\} \) in \( M_{t_0} \).

Next we recall the following corollary of Köbe distortion (= local boundedness of schlicht functions) + Montel:

**Lemma B:** \( B_r := \{ f \in \mathcal{H}(D_r) \mid f(0) = 0, f'(0) = 1, f \text{ is } 1-\text{to-1} \} \) is compact in the normal topology.

By lemma A, \( M_t \) is hyperbolic; so by \( \mathfrak{H} \) there exists a holomorphic isomorphism \( f_t \colon M_t \rightarrow \mathbb{D} \) for each \( t \).

We may assume \( f_t(p_0) = 0 \) for some fixed \( p_0 \in M_t \).

Choose \( t_x \rightarrow \infty \), and denote \( M_{t_x}, f_{t_x} \) by \( M_t, f_t \).

Fix a local coordinate \( z = f_t(p) \) near \( p_0 \), and set

\[
  c_x = f'_t(z(p)) \big|_{p=p_0},
\]

\[
  f_x = \frac{f_t}{c_x} : M_t \rightarrow \mathbb{D}_{c_x},
\]

Write \( N_x \approx \mathbb{Z}_{>0}, \) and (inductively) given \( N_x \subset \mathbb{Z}_{>0} : j \geq 1 \) \( \Rightarrow M_j \subset M_x \)
$F_j'(z(p_0)) = 1, \ F_j \text{ defined/injective on } M_k$

$\Rightarrow F_j \circ F_k^{-1} : D_k \to C \text{ sends } 0 \to 0, \ \text{has degree } = 1$

$\text{by Lemma B: } \exists N_{k+1} \subseteq N_k \text{ s.t. } \{ F_j \circ F_k^{-1} \}_{j \in N_{k+1}} \text{ converges to } H_k : D_k \to C \ (\text{ sends } 0 \to 0, \ \text{with degree } = 1 + 0).$

$(8)$ 

Let $n_j := j^{th}$ entry in $N_j$ ("diagonal subsequence"). Then

$k > 2 \Rightarrow (H_k \circ F_k) \bigg|_{M_k^1} \text{ injective and holomorphic}$

$\text{ by defn. of } H_k$

$$\left( \lim_{j \to n_j} F_j \circ F_k^{-1} \right) \circ F_k = \left( \lim_{j \to n_j} F_j \circ F_k^{-1} \right) \circ F_k \circ F_k^{-1} \circ F_k$$

$$= \left( \lim_{j \to n_j} F_j \circ F_k^{-1} \right) \circ F_k$$

$$= H_k \circ F_k$$

$\Rightarrow H_k \circ F_k = \text{ extension of } H_k \circ F_k \text{ to } M_k$

$k \text{ arbitrary}$

Since $M$ is simply connected, and $f$ is $1$-to-$1$ (\(\Rightarrow\) homeomorphism onto its image), $f(M)$ is simply connected.

Suppose $f(M) \neq C$. By the Riemann Mapping Theorem, we have $M \xrightarrow{\phi} f(M) \xrightarrow{\xi} D$, and clearly $Re(h)$ is a bounded, non-constant harmonic function. By Cor 2 of Lecture 2, $M$ has Green's function, a contradiction. So $f(M) \subseteq C$. 
But the proof is incomplete without

**Lemma C:** All parabolic RSs have a divergent curve.

**Sublemma:** M simply connected RS w/o divergent curve $\Rightarrow$ M\[^p\] simply connected for any p.

**Proof:** As a topological space, M is a connected orientable 2-manifold. Thus M is homeomorphic to a sphere with g > 0 handles and h > 0 points removed. If $g, h \neq (0, 0)$ or $(0, 1)$, then $\pi_1$ is non-trivial. So M is homeomorphic to $\hat{C}$ or $\hat{C'}$. But C has the divergent curve $t(0, 0)$, and thus M is homeo. to $\hat{C}$.

Proof of Lemma C: Given M parabolic w/o divergent curves, M\[^p\] remains simply connected, and clearly has divergent curve (approaching p).

If M\[^p\] is hyperbolic, then $\exists f : M\[^p\] \to D$ (bounded)

- $\exists f : M \to \overline{D}$ extending $f$ with non-zero finite machine.
- $\Rightarrow f : M \to D$ (clearly not 1-to-1, with $d = f(p) = f(q)$)
- OMT $\exists$ disks about $d$ in image of disks about p, q
- $\Rightarrow f$ itself not 1-to-1. $\Box$
So $M(\mathfrak{sp})_\gamma$ must be parabolic. Since it has diverging\namere, $M(\mathfrak{sp})_\gamma \cong \mathbb{C}$ (conformal iso.). Moreover, $M$ \parabolic $\Rightarrow$ noncompact $\Rightarrow$ $M$ \homeomorphic (topological iso.) to $\mathbb{C} \Rightarrow M(\mathfrak{sp})_\gamma$ \homeomorphic to $\mathbb{C}^*$.

So $M$ \parabolic $\Rightarrow M \cong \mathbb{C}$ (conformal iso.).

III. The compact case

Let $M$ be a simply-connected, compact $\mathbb{R}$,$\mathbb{S}$.

$M(\mathfrak{sp})$ noncompact $\Rightarrow M(\mathfrak{sp}) \subseteq \mathbb{C} \cup \mathbb{D}$, by III.1 II.\\
If it's $\mathbb{D}$, Rieman removable sing. $\Rightarrow$ extends to $M \rightarrow \overline{\mathbb{D}}$

$\Rightarrow M(\mathfrak{sp}) \rightarrow \mathbb{D}$ not 1-1. \*\*

So $M(\mathfrak{sp}) \xrightarrow{\cong} F \\mathbb{C}$, and $M$ is the (unique) 1-point\
compensation of $\mathbb{C}$, namely $\mathbb{C}$. 