Lecture 5: Harmonic functions revisited

We now want to take aim at the general Dirichlet problem, solved last term for a disk. To that end, we need a more in-depth understanding of harmonic (and "subharmonic") functions.

I. The mean-value property

Let \( U \) be a domain, \( u: U \rightarrow \mathbb{R} \) a continuous function. Recall that

\[
\text{\( u \in \mathcal{H}(U) \Leftrightarrow u \in C^2_{\mathbb{R}}(U) \) and \( \Delta u \equiv 0 \).}
\]

\[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 4 \pi \xi \frac{\partial^2 u}{\partial z^2}\]

Definition \( u \) has the SCMVP \( \Leftrightarrow \)

\[\forall z_0 \in U, \exists \xi \in (0, d(z_0, U^c)) \text{ s.t.} \]
\[
u(z_0) = \frac{1}{2\pi} \int_{0}^{2\pi} u(z_0 + \xi e^{i\theta}) d\theta \quad \forall \xi \in (0, \varepsilon).
\]
Notice that if $V \subseteq U$ is a subregion,

\[ (*) \quad u \text{ has SCMVP on } U \implies u \text{ has SCMVP on } V. \]

Last term we proved the mean-value theorem for harmonic functions:

**MVT** \[ u \in \mathcal{H}(U) \implies u \text{ has SCMVP.} \]

**Sketch**: on a sufficiently small disk $D = D(z_0, r)$

\[ f \in \mathcal{H}(D) \text{ s.t. } \text{Re}(f) = u. \text{ Then} \]

\[ f(z_0) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)}{z - z_0} \, dz = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{re^{i\theta}} \, rie^{i\theta} \, \text{d}\theta \]

\[ = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) \, d\theta \]

And we can take $\text{Re}$ of both sides.

It turns out that the converse is also true!

**TVM** \[ u \text{ has SCMVP } \implies u \in \mathcal{H}(U). \]

Philosophically, and in terms of how we'll use it, this is a bit like a Morera theorem for harmonic functions. For the proof, we'll need a maximum principle for functions with the SCMVP.
Lemma: Given \( v \in C^0_{kr}(V) \) (\( V \) a region) satisfying SCMVP, and \( p \in V \) s.t. \( v(p) = \sup_{x \in V} v(x) \), we have \( v \equiv \text{Constant} \).

Proof: Set \( \mu := \sup_{x \in V} v(x) \), \( M := \{ x \in V \mid v(x) = \mu \} \).

Clearly, \( p \in M \) (\( \Rightarrow M \neq \emptyset \)). Given \( \{ x_i \} \subset M \) with \( \lim x_i = x_0 \in V \), \( n \in C^0 \Rightarrow (p = \lim x_i) \lim v(x_i) = v(x_0) \Rightarrow x_0 \in M \Rightarrow M \) closed in \( V \).

So if \( M \) is open, then \( M = V \). Consider \( p \in M \); then \( \exists \varepsilon \in (0, d(p, V^c)) \) s.t.

\[
\mu = v(p) = \frac{1}{2\pi} \int_{0}^{2\pi} v(p + r e^{i\theta}) \, d\theta \leq \frac{1}{2\pi} \int_{0}^{2\pi} \mu \, d\theta = \mu \quad (\forall \varepsilon \in (0, \varepsilon))
\]

\( \Rightarrow v(p + r e^{i\theta}) = \mu \quad (\forall \theta, r) \Rightarrow D(p, \varepsilon) \subset M \).

So \( M = V \) and \( v \equiv \mu \) is constant.

Proof of "TVM": Given \( \overline{D} = \overline{D(\zeta_0, \varepsilon)} \subset U \), by the solution to Dirichlet in a disk,

\[
\exists \tilde{u} \in C^0(\overline{D}) \text{ s.t. } \begin{cases} \tilde{u} \in X(D) \\ \tilde{u}|_{\partial D} = u|_{\partial D} \end{cases}
\]

Consider \( v := u - \tilde{u} \in C^0(\overline{D}) \); clearly \( v|_{\partial D} \equiv 0 \),
and \( u \) satisfies the SCMV on \( D \) (using MVT on \( \cdot \)).

Now the Lemma \( \Rightarrow \) if \( u \) attains maximum in \( D \), then \( u \) constant (since identically \( 0 \), since \( C^0 \) and zero on boundary).

So either \( u \equiv 0 \) or \( u < 0 \) in \( D \) \( \Rightarrow \) \( u \leq 0 \) in \( D \).

Apply the same argument to \( -u \) \( \Rightarrow \) \( -u \leq 0 \) in \( D \).

So \( u \equiv 0 \) \( \Rightarrow \) \( u = u^\circ \) \( \Rightarrow \) \( u_0 \in C^0(\partial D) \).

Since \( D \) was arbitrary, \( u \in H(U) \).

**Corollary:** Given \( \{u_j\} \subset H(U) \) converging uniformly on compact subsets of \( U \) to \( u : U \rightarrow \mathbb{R} \), we have \( u \in H(U) \).

**Proof:** Since \( \{u_j\} \subset C^0(U) \), \( u \) is \( C^0 \). Given

\[
\overline{D} = \overline{D}(2r, r) \subset U, \text{ by MVT}
\]

\[
u_j (2r) = \frac{1}{2\pi} \int_0^{2\pi} u_j (2r + re^{i\theta}) \, d\theta
\]

\[
u (2r) = \frac{1}{2\pi} \int_0^{2\pi} u (2r + re^{i\theta}) \, d\theta
\]

\( \Rightarrow \) \( u \) has SCMV \( \Rightarrow \) \( u \in H(U) \).
II. Harnack's principle

Recall the problem from last term's final exam:

Given $u \in H(D_1)$, $u \geq 0$, $u(0) = 1$, prove that $\frac{1}{4} \leq u(\frac{3}{4}) \leq 7$.

This is a special case of

Harnack's inequality (1884)

Let $u \in H(D_R)$ be nonnegative, $z \in D_R$. Then

\[
\frac{R-1|z|}{R+1|z|} u(0) \leq u(z) \leq \frac{R+1|z|}{R-1|z|} u(0).
\]

Remark: If the disk isn't centered at the origin, an obvious corollary (just by shifting everything) is

\[
\frac{R-|z|-2|z|}{R+|z|-2|z|} u(0) \leq u(z) \leq \frac{R+|z|-2|z|}{R-|z|-2|z|} u(0).
\]

Proof: The Poisson formula says

\[
u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(Re^{i\theta}) \frac{R^2 - |z|^2}{|Re^{i\theta} - z|^2} \, d\theta.
\]

We have
\[
\frac{R^2 - |z_1|^2}{|Re^{i\theta} - z_1|^2} \leq \frac{R^2 - |z_1|^2}{(R - |z_1|)^2} = \frac{R + |z_1|}{R - |z_1|}
\]

\[
\frac{R^2 - |z_1|^2}{|Re^{i\theta} - z_1|^2} \geq \frac{R^2 - |z_1|^2}{(R + |z_1|)^2} = \frac{R - |z_1|}{R + |z_1|}.
\]

Since \( u(Re^{i\theta}) \geq 0 \), we can multiply both of these inequalities by \( \cos \theta \). So,

\[
\frac{1}{2\pi} \int_0^{2\pi} u(Re^{i\theta}) \left( \frac{R - |z_1|}{R + |z_1|} \right) d\theta \leq u(0) \leq \frac{1}{2\pi} \int_0^{2\pi} u(Re^{i\theta}) \left( \frac{R + |z_1|}{R - |z_1|} \right) d\theta.
\]

\[
\left( \frac{R - |z_1|}{R + |z_1|} \right) u(0) \leq u(0) \leq \left( \frac{R + |z_1|}{R - |z_1|} \right) u(0).
\]

**Harnack's Principle (1887)**

Let \( U \) be a region,

and \( \{u_j\} \subset H(U) \) a sequence with \( u_1 \leq u_2 \leq \cdots \).

Then \( u_j \to u \) uniformly on compact sets of \( U \).

OR \( \exists \nu \in H(U) \) s.t. \( u_j \to \nu \) uniformly on compact sets.

Remark: So, for example, an increasing sequence of harmonic functions with \( \{u_j(x_0)\} \) bounded for one \( x_0 \in U \), converges to a harmonic function! This seems so surprising that when Harnack told it to Felix Klein, the latter refused to accept its validity! //
Proof:  Set \( U^{\text{fin}} := \{ z \in U \mid \lim u_j(z) < \infty \} \)
\[ U^\infty := \{ z \in U \mid \lim u_j(z) = \infty \}. \]

First suppose \( U^\infty \neq \emptyset \): for \( p \in U^\infty \), \( \exists J \) s.t. \( u_j(p) > 0 \) for \( j \in J \). Clearly \( \exists R \) s.t. \( \overline{D}(p,R) \cap U \) and \( u_j \big|_{\overline{D}} \) (hence every \( u_j \big|_{\overline{D}} \), \( j \in J \)) is positive.

So Hölder's Inequality applies, and for \( z \in D(p,R/2) \)
\[ u_j(z) \geq \frac{R-|z-p|}{R+|z-p|} u_j(p) \geq \frac{R-R/2}{R+R/2} u_j(p) = \frac{u_j(p)}{3} \to \infty \]
and \( u_j(z) \) goes uniformly to \( \infty \) on \( D(p,R/2) \).

Next suppose \( \exists q \in U \) s.t. \( u_j(q) \to 1 < \infty \), i.e. \( q \in U^{\text{fin}} \neq \emptyset \), and let \( \overline{D}(q,\delta) \subset U \). Hölder's Inequality applies to the differences which are nonnegative, so for \( z \in D(q,\delta) \)
\[ 0 \leq u_j(z) - u_j(q) \leq \frac{s + \frac{1}{2} - \frac{1}{4}}{s - \frac{1}{2} - \frac{1}{4}} (u_j(q) - u_j(q)) \]
\[ (j \in \mathbb{N}) \]
\[ \Rightarrow \text{first uniformly Cauchy in } \| \|_{D(q,\delta/2)} \]
\[ \Rightarrow \{ u_j \} \text{ converges pointwise to some function } u \text{ (uniformly in } D(q,\delta/2)) \]
\[ \Rightarrow u \text{ is harmonic on } D(q,\delta/2). \]

(Conduing to Thm)

Conclude thus \( U^{\text{fin}}, U^\infty \) are both open.
Moreover, clearly $U = U^\text{fin} \sqcup U^\infty$, and so $U$ connected $\implies U^\text{fin}$ or $U^\infty$ is empty.

Further, for $K \subset U$ compact, $K$ is covered by a finite collection of balls $D(p, R_k)$ or $D(q, s_k)$ as above; and by uniformity of $u_j \to u_0$ on $K$ on these balls, we get uniform convergence on $K$. $\square$
III. What is... a subharmonic function?

The "harmonic functions" on \( \mathbb{R} \) — i.e., those killed by \( \Delta = \partial_x^2 \) — are just the affine functions

\[
f(x) = ax + b.
\]

On any interval, they clearly satisfy a "maximum principle": if the maximum is achieved in the interior, then the function is constant. If we are after a larger class of functions for which this principle holds, we might consider the convex functions: given any \( a \leq b \), these functions satisfy

\[
g(tb + (1-t)a) \leq tg(b) + (1-t)g(a) \quad \forall t \in [0, 1]
\]

for any affine \( f \) with \( f(a) \geq g(a) \), \( f(b) \geq g(b) \), we have \( g(a) \leq f(x) \) \( \forall x \in [a, b] \).

This definition generalizes easily to a complex variable setting. Notice that if a convex function \( g \) is \( C^2 \), then we can take \( \Delta g = \frac{\partial^2 g}{\partial x^2} \); if this
is $< 0$ at any point, hence on some interval $(a,b)$, we get a function $g_0 = g - f_0$ satisfying
\[
\Delta g_0 < 0 \quad \text{on } [a,b], \quad g_0(a) = g_0(b) = 0.
\]
This is impossible (why?), so we conclude that convex functions which are $C^2$ satisfy $\Delta g \geq 0$.

Now for the complex analogue, which first appeared in work of Poincaré and Hartogs, and was then systematically studied by Riesz in the 1920s. Let $U \subset \mathbb{C}$ be open, $f \in C^2(U)$.

**Definition**  
$f \in \mathcal{H}(U)$ (f is subharmonic on $U$) 

\[\forall z_0 \in U, \ r \in (0, \delta(z_0, U^c)), \ u \in \mathcal{H}(\overline{B}(z_0, r)) \]

satisfying $f \leq u$ on $\partial B(z_0, r)$, we have $f \leq u$ on $D(z_0, r)$.

Suppose $h \in \mathcal{H}(U)$. Is $h$ subharmonic?? Well, let $\overline{D} = \overline{B}(z_0, r) \cap U$ and $u \in \mathcal{H}(\overline{D})$ be such that $h \leq u$ on $\partial \overline{D}$, i.e. $h - u \geq 0$ there. By the maximum
principle for harmonic functions, $h - u \leq 0$ on $\overline{D}$. So $H(U) \leq H(U)$.

Remark: One can define superharmonic functions $\overline{H}(U)$ by reversing the inequalities in the above definition.

The following is an analogue of the MT for subharmonic functions, and is very useful for constructing them.

**Theorem** Let $f \in C^0_{\overline{D}}(U)$, $U \subset \mathbb{C}$ open. Then $f \in \overline{H}(U) \iff f(p) \leq \frac{1}{2\pi} \int_{0}^{2\pi} f(p + re^{i\theta}) d\theta \quad \forall \overline{D}(p, r) \subset U$.

Proof: Suppose ($\ast$) holds. If $f \notin \overline{H}(U)$, then $\exists \overline{D}(\overline{z}, s) =: \overline{D}' \subset U$ and $h \in H(\overline{D}')$ s.t. $f \leq h$ on $\partial \overline{D}'$ but $f(z_0) > h(z_0)$ for some $z_0 \in \overline{D}'$. Consider $g := f - h$ on $\overline{D}'$, so that $\{ g \leq 0 \text{ on } \partial \overline{D}' \}$ and $\{ g(z_0) > 0 \}$. Let $M = \max_{\overline{D}'}(g)$, $K = \{ z \in \overline{D}' \mid g(z) = M \} \subset \overline{D}'$. Compact.

If we $\exists k$ then $\exists z \in \partial \overline{D}(w, r)$ with $g(z) < M$. ($\text{Here } \overline{D}(w, r) \subset U$)
Since $g$ is $C^0$, there is a whole arc of $\partial D(w, \epsilon)$ where $g < M$, so

$$\frac{1}{2\pi} \int_0^{2\pi} g(w + \epsilon e^{i\theta}) d\theta < M > g(w)$$

[\text{by MVT}]

$$\frac{1}{2\pi} \int_0^{2\pi} f(w + \epsilon e^{i\theta}) d\theta - h(w)$$

$$\Rightarrow f(w) \leq \frac{1}{2\pi} \int_0^{2\pi} f(w + \epsilon e^{i\theta}) d\theta < h(w) + g(w) = f(w),$$

a contradiction.

To do the converse, suppose $f \in \mathcal{H}(U)$, and fix $\overline{D}(x, r) =: \overline{D} \subset U$. Let $P: D \times D \rightarrow \mathbb{R}$ be the Poisson kernel $P(z, e^{i\theta}) = \frac{1}{2\pi} \frac{1 - |z - e^{i\theta}|^2}{|z - e^{i\theta}|^2}$ for $D$. Then $h(z) := \frac{1}{2\pi} \int_0^{2\pi} P(z, e^{i\theta}) \cdot \{r(z + \epsilon e^{i\theta}) + \epsilon\} d\theta$ defines a continuous function on $\overline{D}$ which is harmonic on $D$.

Moreover, for any $z \in D$, $h(z) = f(z) + \epsilon > f(z)$.

By continuity of $f \upharpoonright h$, $h(z) > f(z)$ for $z \in D(z, \delta)$ for $\delta > 0$ ($\delta$ depending on $\epsilon$) small. But then by subharmonicity, $f \leq h$ on $D(z, \delta)$, hence
\[
\mathcal{H}(\mathbb{C}) \leq \mathcal{H}(\mathbb{C}) = \{ \frac{1}{2\pi} \int_0^{2\pi} \{ f(\theta + se^{i\theta}) + \epsilon \} d\theta \mid \epsilon \geq 0 \}
\]

Take $\epsilon \to 0^+$, done.

**Corollary** \[ \mathcal{H}(\mathbb{C}) = \mathcal{H}(\mathbb{C}) \cap \mathcal{H}(\mathbb{C}) \]

**Proof**: \( \leq \) is obvious. By the theorem, for every small circle we have \( f \in \mathcal{H}(\mathbb{C}) \)

\[
\mathcal{H}(\mathbb{C}) \overset{\frac{1}{2\pi} \int_0^{2\pi} f(p + r e^{i\theta}) d\theta}{\leq} \text{ and } \geq
\]

By \( \text{TVM} \), \( f \in \mathcal{H}(\mathbb{C}) \).

We'll exploit the theorem a great deal more in the next lecture.
IV. A note on Schwarz triangle function

This is about a special case of an exercise in problem set #2, concerning the map

$$F(w) = e^{i	heta} \int_0^w \frac{d\tilde{w}}{\tilde{w}^{2/3} (\tilde{w}-1)^{2/3}} + C$$

These allow an arbitrary rotation and translation.

Sending $h$ to an equilateral triangle in a conformal equivalence. Let

$$K := \text{side length} = \left| \int_0^1 \frac{d\tilde{w}}{\tilde{w}^{2/3} (\tilde{w}-1)^{2/3}} \right|$$

We shall determine the periods of $F^{-1}$ by repeated Schwarz reflection. (The domain for these "reflections" alternates between $h$ and $-h$.)
• first period: \( F(w) \xrightarrow{(a)} S_6 F(w) \xrightarrow{(b)} S_3 F(w) \xrightarrow{(c)} S_6 S_3 F(w) = S_3 F(w) \) 
  \( \xrightarrow{(d)} \) 
  \( \sqrt{3} K_i \) 
  \( = F(w) + \sqrt{3} K_i \) 

(The point is that each “reflection” is also an analytic continuation of \( F^{-1} \), so the formulas prove that \( F^{-1}(w + \sqrt{3} K_i) = F^{-1}(w) \).)

• second period: \( F(w) \xrightarrow{(a)+(b)} S_3 F(w) \) (as above) 
  \( \xrightarrow{\ } \) 
  \( \frac{1}{(S_3 F(w) - S_3 K f_3) S_2 + K} \) 
  \( = S_3 F(w) - S_3 K f_3 + K \) 
  \( \xrightarrow{\ } \) 
  \( \frac{(S_3 F(w) - S_3 K f_3 f_6) S_6 + K}{(S_3 F(w) - S_3 K f_3) f_6} \) 
  \( = F(w) + K(1 + S_6) \) 
  \( \xrightarrow{\ } \) 
  \( F(w) + \sqrt{3} K e^{i\pi/6} \) 

So the period lattice is
\( \Lambda = \mathbb{Z}\langle \sqrt{3} K_i, \sqrt{3} K e^{i\pi/6} \rangle \)
or, after multiplying $F$ by the constant $\frac{1}{\sqrt{3}Ke^{i\pi/6}}$

$$\Lambda \approx \mathbb{Z}\langle 1, e^{i\pi/3} \rangle.$$

The fundamental region for $\mathbb{C}/\Lambda$ is made up of 6 triangles and can be thought of either as

Topologically $\mathbb{C}/\Lambda$ is a torus and is isomorphic to the quartic curve

$$y^3 = w^2(w-1)^2$$

(once you resolve its two singularities and compactify it).

This curve has an automorphism $(y, w) \mapsto (\zeta_3 y, w)$, which corresponds to the "$(\varphi)+(\psi)$" transformation above

$$F(w) \mapsto \zeta_3 F(w).$$
that is, the automorphism of $C/\mathbb{L}$ given by

$$u \ (\text{mod} \ \mathbb{L}) \mapsto \varepsilon_3 u \ (\text{mod} \ \mathbb{L})$$

which is well-defined because $\varepsilon_3 \mathbb{L} \subseteq \mathbb{L}$. We say that $C/\mathbb{L}$ is a curve with complex multiplication; the corresponding (elliptic) algebraic curves are used extensively in cryptography.

Summing up: we have

$$F^{-1}(u + \Lambda) = F^{-1}(u) \ \forall u \in C, \ \forall \Lambda \in \mathbb{L}$$

$\implies F^{-1}$ yields a well-defined function $C/\mathbb{L} \to \mathbb{P}^1$, and also

$$F^{-1}(\varepsilon_3 u) = F^{-1}(u).$$