Lecture 8: Multiply connected regions

Generalizing the result at the end of Lecture 7, we now discuss canonical mappings and conformal invariants for regions with connectivity \( n \geq 1 \), following Ahlfors rather closely. The two main tools are harmonic measures and Green's functions. We begin with a (partial) reformulation of what was proved regarding the Dirichlet problem:

**Theorem 1** Let \( \mathcal{L} \subseteq \mathbb{C} \) be a region, with \( \mathbb{C} \setminus \mathcal{L} \) having a finite number of connected components, none of which are points. Then the Dirichlet problem is solvable for \( \mathcal{L} \).

**Proof:** Write \( \mathbb{C} \setminus \mathcal{L} = E_1 \sqcup \ldots \sqcup E_n \), with \( E_n \) the unbounded component. By the RMT, there exists a conformal equivalence \( \phi_n : \mathbb{C} \setminus E_n \to D_1 \). Set \( D_n := \phi_n(\mathcal{L}) \), \( E_{kn} := \begin{cases} \mathbb{C} \setminus D_1, & k = n \\ \phi_n(E_k), & k < n \end{cases} \).
Next, consider $E_{n-1,n}$, which is simply connected, so RMT $\Rightarrow \exists \phi_{n-1} : \overline{E_{n-1,n}} \sim \overline{D_2}$; we set $\phi_{n-1} : \overline{E_{n-1,n}} \sim \overline{D_2}$.

Iterating this last step for $n-2, n-3, \ldots, 1$ we see that for $k \geq 2$, $\partial E_{k,1}$ is real-analytic Jordan by virtue of being the image of $\partial D_2$ by a composition of conformal 1-to-1 maps. In particular, $\partial R_2 = \bigcup_{i=1}^{\infty} C_i (\equiv \partial E_{1,1})$ is a union of such curves and so $R_2$ has a barrier at any point of $\partial R_2$. Pulling these barriers back by $\phi_1 \circ \cdots \circ \phi_n : \Sigma \to \overline{R_2}$ (extend by Cauchy/Pontryagin) shows the Dirichlet problem is solvable for $\Sigma$.

Let $\Sigma_0$ be the image of $\Sigma$, via $\varepsilon \to \frac{1}{\varepsilon}$.

In what follows it is convenient to work with $\Sigma_0 : = \Sigma_0$ rather than $\Sigma$ for some purposes. Write $\partial R = \Sigma \partial_1 R$. 

\[ \begin{array}{c}
\text{\includegraphics{diagram}}
\end{array} \]
I. Harmonic measures

Definition 1: The harmonic measure \( \omega_{k} \) with respect to \( \Omega \) is the solution to the D.P. with \( f = \begin{cases} 1 & \text{on } \partial_{c} \Omega, \\ 0 & \text{on } \partial_{s} \Omega, \end{cases} \) for \( k \). Call this \( \omega_{k} \in C(\Omega) \).

Properties:

1. Partition of unity: \( \sum_{k=1}^{n} \omega_{k} = 1 \).
   
   Proof: Use max-min principle for harmonic functions.

2. Continuous extension to boundary: \( \omega: \overline{\Omega} \to [0,1] \)
   
   \( \nu \to [0,1] \)
   
   \( \omega_{k} \to (0,1) \)

   Proof: Range is confined to \([0,1]\) again by max-min principle.

Now we restrict to the "smooth boundary case" - i.e. work with \( \overline{\Omega} \).

3. Schwarz extension: Using technique of reflection above an analytic arc (when the harmonic function has \( C^{1} \) extension to this arc and is \( O \) for constant on the arc),
   we obtain \( \tilde{\omega}_{k} \in C(\overline{\Omega}) \) - i.e. a harmonic extension
   on a neighborhood of \( \overline{\Omega} \), with \( \tilde{\omega}_{k}|_{\partial \Omega} = \omega_{k} \).
   (This is false for \( \Omega \) - it would violate maximum on a region misses)
\textbf{Definition 2} The \underline{periods} of $\Sigma$ are the real numbers
\[ \alpha^i_k := \int_{C_k} \star dw_j, \quad k, j = 1, \ldots, n-1. \]

\begin{enumerate}
\item \underline{Conformal invariance:} If $\Sigma \cong \Sigma'$ conformally, then after reordering indices so that the $d_\Sigma \Sigma$ and $d_{\Sigma'} \Sigma'$ match up, $\alpha^i_k = \alpha^j_l \ (\forall j, k)$. \hfill \Box
\end{enumerate}

\textbf{Proof:} Get conformal isom. $\Sigma \rightarrow \Sigma'$, pull everything back — pullback of harmonic by holo. is harmonic, and $d_x \star$, \& periods are all invariant under holo. pullback.

\begin{enumerate}
\item \underline{Linear independence:} The vectors \( \left\{ \frac{\partial}{\partial w_j} \right\}_{j=1, \ldots, n-1} \) are a basis.
\end{enumerate}

\textbf{Remark / For what?} Well, we have bases
\[ \{ \mathcal{C}_k \}_{k=1}^{n-1} \cong H^1(\Sigma, \mathbb{R}) \]
and
\[ \{ \mathcal{C}_k \}_{k=1}^{n-1} \cong H^1(\Sigma, \mathbb{R}) \]
while
\[ H^1(\overline{\Sigma}, \mathbb{R}) \cong H^1(\overline{\Sigma}, \mathbb{R}) = \text{closed real 1-forms} \]
\[ \text{exact real 1-forms} \]

Further, the vector \( \left( \begin{array}{c}
\frac{\partial}{\partial w_1} \\
\vdots \\
\frac{\partial}{\partial w_{n-1}}
\end{array} \right) \) expresses $d \omega$ with respect to the basis $\{ \mathcal{C}_k \}$, so the above result says that $\{ \star dw_j \}_{j=1}^{n-1}$ is a basis for de Rham cohomology. $\Box$
Proof: Suppose $\sum \lambda_j x_j = 0$ for some $\{\lambda_j\} \subset \mathbb{R}$, i.e. $\mathbf{x}^T \mathbf{A} (\sum \lambda_j x_j) = 0$, has trivial solutions. Then

$$f(z) = \sum \lambda_j x_j + i \int x^T (\sum \lambda_j x_j) \in \mathbb{R}(\mathbb{C})$$

is well-defined, with $Re(f) |_{C_l} = \sum \lambda_j, j = 1, \ldots, n - 1$

Assume $f$ nonconstant, so there $\exists w_0$ with $Re(w_0) \neq \{0\}$ or $\lambda_k$, and $z_0 \in \mathbb{C}$ with $f(z_0) = w_0$ (using OMT). By the argument principle

$$0 < \oint_{C_l} darg(f(z) - w_0) = \sum \lambda_j \oint_{C_l} darg(z - w_0)$$

$$= \sum \lambda_j \oint_{f(C_l)} darg(w - w_0), \quad (\#)$$

where $f(C_l)$ is contained in a vertical strip on $Re(w) = \{ \lambda_k, k \neq n \}$. But then $(\#) = 0$, a contradiction. So $f$ is constant, and all $\lambda_k = 0$. \(\square\)

\(6\) Symmetry: $x^T A x = x^T \overline{x}$. 

Proof: Exercise using Theorem 19 of Chap. 4 of Ahlfors. \(\square\)

Now we shall use property \(5\) to prove...
Theorem 2 \[ \exists \phi : \mathbb{R} \to A(1, r) \setminus \bigcup_{i=1}^{n} C_i(\theta_i^-, \theta_i^+) \]

Proof: Write \[ \eta = [\eta_i^j] \text{ (a } (n-1) \times (n-1) \text{ matrix) } \]

By (5) there is a solution to \[ \sum_j A_{ij} \eta_j = (2\pi, 0, \ldots, 0) \]

and taking \( f = \sum_{j=1}^{n} A_{ij} \eta_j + i \sum_{j=1}^{n} \chi_j \phi dw_j \)

(\( \phi \) multivalued function with branches \( 2\pi i \) about \( C_i \) and \( -2\pi i \) about \( C_i \))

we find \( F = e^f \in \mathcal{H}(\mathbb{D}) \) (using property (3)).

Clearly \[ |F| \big|_{C_k} = \begin{cases} e^{2\pi i} & \text{if } k \neq n \\ 1 & \text{if } k = n \end{cases} \]

and we put \( r_i = e^{2\pi i} (i = 1, \ldots, n-1) \). Since \( \sum_i r_i = \sum_i C_i \), by the argument principle together with the fact that \( \arg(F(z)) \)
drops by \( \pi \) when \( w \) passes over \( w_0 \),

\[ \frac{1}{2\pi i} \sum_{i=1}^{n} \int_{C_i} \frac{1}{F(z) - w_0} \, dz = \sum \{ z \in \mathbb{C} \mid F(z) = w_0 \} + \frac{1}{2} \sum \{ z \in \partial \mathbb{D} \mid F(z) = w_0 \} \]

where both "#s" are interpreted with multiplicity.

To compute LHS(\( \eta \)), use the fact that the \( F(C_i) \) are circular arcs. For \( w, w_0 \) on the same circular arc,
\[
\text{We have: } \text{arg} (w - w_0) = \frac{1}{2} \text{arg} (w).
\]

We know that \( \text{arg} (F(\mathbb{D})) = \mathfrak{F}_n(\mathbb{D}) \) has periods \( \{ \pm 2\pi \} \) about \( C_1 \) and \( C_n \), and \( 0 \) about \( C_2, \ldots, C_{n-1} \), by construction.

So (a) \( F(C_1) \) and \( F(C_n) \) are full circles.

(b) \( C_1 \) and \( C_n \) are mapped in 1-1 fashion on these circles, because LHS (\( \mp \pi \)) is \( w \in F(C_1) \), so \( \text{RHS} (\mp \pi) \) is \( w + \frac{\pi}{2} \cdot 1 \).

(c) \( C_2, \ldots, C_{n-1} \) are mapped to a single back-and-forth along a circle, as \( w \in F(C_1) \), \( w \in F(C_2), \ldots, w \in F(C_{n-1}) \) yields 1 on LHS (\( \mp \pi \)), which on RHS (\( \mp \pi \)) must be interpreted as \( 0 + \frac{\pi}{2} \cdot 2 \) (as \( 1 + \frac{\pi}{2} \cdot 0 \) is impossible).

In fact, this numerical condition shows that \( F(\mathbb{D}) \cap F(\mathbb{D}) \) is empty, and since \( F(\mathbb{D}) \) is connected, this is why \( F(C_2), (1 = 2, \ldots, n-1) \) must be contained in \( \mathbb{D} \) (open connected closed subsets of circles).
By uniqueness of the choice of $F$ (up to $f \circ f \circ C \Rightarrow F \rightarrow F \cdot K$), we get uniqueness of the image region $F(\vec{r})$ up to rotation. (Possible dilations are taken care of by the normalization of the inner radius.) This gives the

**Corollary** An $n$-connected region $\mathcal{R}$ has

\[
1 + 3(n-2) - 1 = 3n - 6
\]

real moduli; i.e., is determined up to conformal equivalence by the point

\[
\left[ (r_1, r_2, \ldots, r_{n-1}; \theta_2, \ldots, \theta_{n-1}; \theta_2^b, \ldots, \theta_{n-1}^b) \right] / \text{rotational}
\]

in $\mathbb{R}^{n-1} \times (\mathbb{R}/2\pi\mathbb{Z})^{2n-4}$.

One might ask: is this point determined by the periods $\{a_\ell\}$? In this case they would finish a complete set of conformal invariants. While we'll see in §III that this follows from a result in algebraic geometry, it would be interesting to have a direct proof.

For now, that this should be true is strongly suggested by the following table:
\[
\begin{array}{|c|c|c|}
\hline
n \text{ (connectivity)} & \binom{n}{2} \text{ (all of independent)} & \begin{cases}
3n-6, n > 2 \\
1, n = 2 \text{ (product)}
\end{cases} \\
\hline
2 & 1 & 1 \\
2 & 3 & 3 \\
4 & 6 & 6 \\
5 & 10 & 9 \\
6 & 15 & 12 \\
\vdots & \vdots & \vdots \\
\hline
\end{array}
\]

Which also says that there eventually appears a \underline{locus} in the space of periods — the period \(\binom{n}{2}\)-tuples that “can come from a region.” What is this locus? I don’t know if this is totally understood. These two questions are “real” instances of the \underline{Torelli theorem} and \underline{Schottky problem} for \underline{algebraic curves} (a.k.a. \underline{Riemann surfaces}). Here “real” refers to the fact that the \underline{differentials /periods} in the latter case are \underline{complex}, so the situation is a bit different.
II. Green's function

[Definition 3] Given a region \( \Omega \subset \mathbb{C} \), a Green's function of \( \Omega \) with singularity at \( z_0 \) is a function

\[
g(z; z_0) : \Omega \setminus \{z_0\} \rightarrow \mathbb{R}
\]

with the properties:

(a) \( g(z; z_0) \) is harmonic in \( \Omega \setminus \{z_0\} \)

(b) \( G(z) := g(z; z_0) + \log |z - z_0| \) is harmonic in a disk about \( z_0 \)

(c) \( \lim_{z \to z_0} g(z; z_0) = 0 \) \( \forall z \in \partial \Omega \).

Properties:

1. Uniqueness: Given \( g, \tilde{g} \) satisfying (a)-(c) for \( \Omega, z_0 \), (a)-(b) \( \Rightarrow g - \tilde{g} \in H(\Omega) \)

\[
\left\{ \begin{array}{c}
g - \tilde{g} = 0. \\
g - \tilde{g} \to 0 \text{ as } \partial \Omega
\end{array} \right\}
\]
(2) **Existence:** If \( \Omega \subset \mathbb{C} \) is bounded, and the D.P. is solvable in \( \Omega \), then for each \( \varepsilon \in \Omega \) Green's function: just solve D.P. for \( \log|z-\varepsilon| \) with \( G(z) \), then subtract off \( \log|z-\varepsilon| \).

*Remark:* Green's function does NOT exist for, say, \( \Omega = \mathbb{C} \). (The problem, naturally, is the lack of a conformal mapping into a banded region.) I'll leave it as an exercise — use the following property:

(3) **Positivity:** \( g > 0 \) on \( \Omega \setminus \{ \varepsilon \} \).

Clearly this is true sufficiently close to \( \{ \varepsilon \} \), since \( \log|z-\varepsilon| + g(z, \varepsilon) \) is banded. Applying (c) and max/min on \( \Omega \setminus D(\varepsilon, \varepsilon) \) does the job.
4) Invariance: If \( \phi: \mathbb{R} \rightarrow \mathbb{R} \)

\[ (z_0 \rightarrow z_0') \]

is a conformal isomorphism, then \( g(\phi(z), \phi(z_0)) = g(z, z_0) \) — i.e., the pullback of a Green's function yields a Green's function.

**Sketch**: The main point is that \( \log |z-z_0| = (z-z_0)H(z) \), H holo.

**Proof**: The main point is that \( \log |z-z_0| = (z-z_0)H(z) \), H holo.

5) Symmetry: \( g(z_1, z_2) = g(z_2, z_1) \).

\[ (\Rightarrow \text{ harmonic off } z_1 = z_2 \text{ in both variables}) \]

Proof: We use the key result from last term that for any two harmonic functions \( h_1, h_2 \) on a region,

\[ h_1 \partial h_2 - h_2 \partial h_1 \text{ is a closed 1-form}, \]

hence has periods depending only on the homology class of the path. But writing (on \( D(z_1, z_2) \))

\[ g(z) = g(z, z_1) (z \rightarrow z_2) \text{ and } g_1 + g_2 \equiv 0 \text{, } \]

we have
\[ \int_{\gamma_1} g_1 \, dg_2 - g_2 \, dg_1 = \int_{\gamma_2} g_1 \, dg_2 - g_2 \, dg_1 = 0 \]

\[ \Rightarrow \quad \int_{\gamma_1} g_1 \, dg_2 + \int_{\gamma_2} g_1 \, dg_2 = \int_{\gamma_1} g_2 \, dg_1 + \int_{\gamma_2} g_2 \, dg_1 \]

Taking limits as \( \gamma_1 \) and \( \gamma_2 \) shrink, the closed terms go to 0 (as \( \epsilon \to 0 \)) and \( \epsilon \) \( \text{d}g_2 \); in the other two terms, it reduces to \( \log (e - \epsilon) = \epsilon \).

\[ \Rightarrow \quad 2\pi g_1(\epsilon, \lambda) = 2\pi g_2(\lambda) \]

**6. Relation to \( \omega_k \):**

\[ \frac{1}{2\pi} \int_{C_k} x \, d\gamma (e, z_0) = \omega_k (z_0) \]

for any \( z_0 \in \mathbb{R} \).

**Proof:**

\[ 2\pi \omega_k (z_0) \overset{\text{MVT}}{=} \lim_{\epsilon \to 0} \int_{C_k} \omega_k + dg_2 - g_2 \, d\omega_k \]

\[ \left( C_k(\epsilon, x_0) \equiv [a, b] \right) \]

\[ (g = 0 \text{ on } \partial \mathbb{R}) \]

\[ (\omega_k = 0 \text{ on } \partial \mathbb{R} \backslash C_k) \]

\[ \int_{\partial \mathbb{R}} g \, d\gamma = \int_{C_k} g \, d\gamma \]
The main application of all this is to obtain a different sort of canonical mapping, from \( D \) to \( \mathbb{C} \setminus \{ n \text{ vertical strips} \} \). Using the fact that locally a harmonic function is \( \text{Re}(\zeta) \) hence has harmonic partial derivatives, Ahlfors obtains the

**Lemma:** If \( g = \text{Green's function for } \mathbb{D} \) then

\[
u_0(z) := \frac{\partial}{\partial x_0} g(z, z_0) \text{ is } \begin{cases} \text{harmonic on } \mathbb{D} \setminus \{ z_0 \} \\ \text{zero on } \partial \mathbb{D} \end{cases}
\]

and differs from \( \text{Re} \left( \frac{1}{z - z_0} \right) \) by a harmonic function.

Now, setting \( \Delta_k := \int_{C_k} \), we can use the \((\omega_k)\) to kill these parts; i.e., there is

\[
u := \nu_0 + \lambda_1 \omega_1 + \cdots + \lambda_{n-1} \omega_{n-1}
\]

such that

\[
\text{supplies the } \text{Re} \left( \frac{1}{z - z_0} \right) \quad \text{“supplies the } \text{Im} \left( \frac{1}{z - z_0} \right)
\]

\[
f = \nu + \int \text{Re} \, dz_0 \in H^0(\mathbb{D} \setminus \{ z_0 \})
\]

with a simple pole at \( z_0 \) with residue 1.

Moreover, since \( \nu_0 |_{C_k} \equiv 0 \) and \( \omega_k |_{C_k} = \delta_k \),
we have \(\nu|_{C_0} = \lambda_0\). So the \(C_0\) are mapped to slits along \(\text{Re}(z) = \lambda_0\).

Exercise: Which annulus does the region \(\{e^{\frac{1}{2}\pi i z} \}^{2 \leq \pi \leq 3}\) map to? (Of course, it will depend transcendently on the locations & sizes of the slits!)
III. The Schottky double

Before leaving multiply-connected regions, I want to give a beautiful relation between them and compact Riemann surfaces (complex 1-manifolds) which is apparently due to Schottky & Spenar in the form I'll give it.

Any given $n$-connected domain as in Thm. 1 is biholomorphic to the complement in $\mathbb{C}$ of $n$ bounded simply connected regions with analytic Jordan boundary:

\[ \beta_0, \beta_1, \beta_2, \ldots, \beta_n = \Sigma \]

Write $g := n - 1$, \[ \partial g = \sum_{j=0}^{g} \beta_j. \] As before, we have the harmonic measures

\[ \omega_j (z) = \frac{1}{2\pi} \int_{\beta_j} \frac{1}{|z - \gamma|^2} d\gamma, \quad j = 1, \ldots, g, \]
where $\omega_j \mid \phi_k \equiv \delta_{jk}$ ($k = 0, \ldots, g$). Furthermore we had the periods $\Pi \phi_k := \oint \frac{d\omega_k}{\phi_k}$, $k = 1, \ldots, g$, which were shown to constitute a matrix of maximal rank and which you will show is symmetric.

Assuming that $0 \notin \hat{\Sigma}$, $\frac{1}{2}$ is a holomorphic coordinate on $\hat{\Sigma}$ vanishing only at $\{0\}$. As a set, the Schottky double is

$$\Sigma = \hat{\Sigma} \sqcup \hat{\Sigma} \sqcup d\hat{\Sigma}$$

where we declare $\frac{1}{2}$ to be the hole and $\frac{1}{2}$ to be the hole coordinate to get a complex analytic structure on $\Sigma \setminus 2\hat{\Sigma}$. To put an analytic structure on neighborhoods of points of $2\hat{\Sigma}$, we declare that a meromorphic function on $\Sigma$ is a pair of meromorphic functions $f, \bar{f}$ on $\hat{\Sigma}$ such that $f(\bar{e}) = \bar{f}(\bar{e})$ on the boundary. A meromorphic 1-form on $\Sigma$ is a pair of meromorphic 1-forms $f \bar{\omega} + \bar{f} \omega$ which agree on the boundary. Locally, this gives you something like the picture.
or more globally (say, with $n=4$)

where the $\{\beta_j\}$ are as before, and the $\{\gamma_j\}$ are
defined by fixing points $\beta_0, \beta_1, \ldots, \beta_g$ on each boundary component, and going from $\beta_0$ to $\beta_j$ on the holomorphic sheet then from $\beta_j$ to $\beta_0$ on the antiholomorphic sheet.

We have $\mathcal{H}_g(\Sigma, \mathbb{Z}) \cong \mathbb{Z}\langle \gamma_1, \ldots, \gamma_g; \beta_1, \ldots, \beta_g \rangle$ with intersection matrix

\[
\begin{pmatrix}
0 & -I_g \\
I_g & 0
\end{pmatrix},
\]

and $g$ is the genus of $\Sigma$.

What about holomorphic forms to integrate over these cycles? Set $W_j := \frac{1}{2}(d\omega_j + i\bar{d}\omega_j)$ on the holomorphic sheet; this has zero real part $d\omega_j$ on each $\beta_i$ (since $\omega_j$ is constant there). Consequently,

\[
\eta_j := \begin{cases}
W_j & \text{on hol. sheet} \\
-\bar{W_j} & \text{on antihol. sheet}
\end{cases}
\]

yield holomorphic 1-forms on $\Sigma$. Further, their periods are

\[
\oint_{\beta_i} \eta_j = \int_{\gamma_i} W_j - \int_{\gamma_i} \bar{W_j}
\]

\[
= \int_{\gamma_i} \text{Re}(W_j) = \int_{\gamma_i} d\omega_j
\]

\[
= \delta_{ij}
\]
and \[ \phi_{\beta_1} \eta_{\epsilon} = \frac{i}{2} \phi_{\beta_2} d\omega_{\epsilon} \]
\[ = \frac{i}{2} \Pi_{ke} . \]

In general for Riemann surfaces, one has the result that there are always bases for \( H_2 \) and the holomorphic 1-forms yielding the matrix
\[
\begin{pmatrix}
\Pi_g \\
\frac{1}{2}
\end{pmatrix}
\]

where

- \( \frac{1}{2} = \epsilon \frac{Z}{2} \) and
- \( \text{Im}(Z) > 0 \).

By a famous result of Torelli, the matrix \( Z \) completely determines, up to change of basis for \( H_2 \), the (complex analytic) isomorphism classes of Riemann surfaces in a given family. In our case,

\[ Z = \frac{8}{2} \Pi . \]
and so the periods \( \{ T_{1, k} \} \) determine the conformal isomorphism class of a given \( \Sigma \) with fixed ordering of the boundary components. However, the proof of Torcelli's theorem is well beyond the scope of this course.