Problem set 10

(1) Find polynomials (for each \( n \)) realizing the upper bound (namely, \( \frac{n}{n+1} \)) on \( \sigma(P) \) in the Smale Conjecture.

(2) Examine the proof of Bloch’s theorem to prove that \( L \geq \frac{1}{24} \).

(3) Prove that the power-series coefficients \( a_n \), viewed as functions on the set \( S \) of schlicht functions, are continuous functions in the normal topology. (This completes the argument indicated in class that there must exist universal upper bounds on them.)

(4) Let \( \Omega \subsetneq \mathbb{C} \) be a simply connected domain, \( \alpha \in \Omega \) a point. The RMT guarantees a conformal isomorphism \( F: \Omega \to D_1 \) with \( F(\alpha) = 0 \). Show that \( \frac{1}{4d(\alpha, \partial \Omega)} \leq |F'(\alpha)| \leq \frac{1}{d(\alpha, \partial \Omega)} \). (Don’t use anything beyond Köbe \( \frac{1}{4} \) in the notes.)

(5) Prove that the universal cover of \( \mathbb{C}^* \setminus [1, \infty) \) cannot be \( \mathbb{C} \) or \( \mathbb{P}^1 \).

(6) If \( \Gamma \) is the fundamental group (the group of homotopy classes of loops) of a Riemann surface \( M \) having \( \mathcal{H} \) as universal covering surface, then \( M \cong \mathcal{H}/\Gamma \). Prove that 2 Riemann surfaces \( M_i \cong \mathcal{H}/\Gamma_i \) \((i = 1, 2)\) are isomorphic if and only if \( \Gamma_1 \) and \( \Gamma_2 \) are conjugate in \( \text{PGL}(2, \mathbb{R}) \).

(7) Let \( S^* := \{ f \in S \mid f(D) \text{ is starlike} \} \). (A set \( E \) is starlike if for every \( z \in E \), \( tz \in E \) for every \( t \in [0, 1] \).) Prove that \( \text{Re}(z^{L_F}) \geq 0 \) for an \( f \in S^* \) by following the steps:
   
   (a) Show \( \frac{\partial}{\partial \theta} \text{arg}(f(re^{i\theta})) > 0 \), for any \( r \in (0, 1) \). [Hint: first show that \( f(\overline{D}_r) \) is starlike, making use of the function \( f^{-1}(rf(z)) \) and Schwarz’s Lemma.]
   
   (b) Show that \( \frac{\partial}{\partial \theta} \text{arg} f = \text{Re}(z^{L_F}) \), for \( |z| = r \in (0, 1) \).

(8) Again let \( f \in S^* \), and define \( g(z) := \int_0^z \frac{f(w)dw}{w} \in \text{Hol}(D) \).
   
   (a) Show that \( \text{Re}(1 + z\frac{g'}{g}) \geq 0 \). [Hint: use problem 3.]
   
   (b) Prove \( g(D) \) is convex. [Hint: show \( \frac{\partial}{\partial \theta} \text{arg}(\frac{\partial}{\partial \theta} g(re^{i\theta})) \geq 0 \), and interpret this geometrically.]
   
   (c) Writing \( g(z) = z + \sum_{n \geq 2} b_n z^n \), show that each \( |b_m| \geq 1 \). [Hint: define \( G(z) := \frac{1}{m} \sum_{j = 1}^m g(\zeta_m z) \), where \( \zeta_m = e^{2\pi i/m} \). Explain why \( G(D) \subset g(D) \), then consider \( G(z) := g^{-1}(G(z)) \) and compare the lowest order terms in the expansions of \( g(h(z)) \) and \( G(z) \).]
   
   (d) Writing \( f(z) := z + \sum_{n \geq 2} a_n z^n \), deduce that \( |a_m| \leq m \) for each \( m \).