Problem Set 12

(1) Let $f : M \to N$ be a map of Riemann surfaces. In each case prove that f is constant:
(a) $M \cong \mathbb{P}^1$ and $N \ncong \mathbb{P}^1$
(b) $M \cong \mathbb{C}$ and $N \ncong \mathbb{P}^1, \mathbb{C}, \mathbb{C}^*, \text{or } \mathbb{C}/\Lambda$.

(2) Demonstrate that $U(n)$ is nonabelian for $n \geq 2$.

(3) Suppose you are given a domain $\Omega \subset \mathbb{C}^n$ and a holomorphic map $F = (f_1(\underline{z}), \ldots, f_n(\underline{z}))$ from $\Omega \hookrightarrow \mathbb{C}^n$ sending $\underline{0}$ to $\underline{0}$. Furthermore, suppose that the k-fold composition, which will in general only be defined on a smaller domain $\Omega_0 \subset \mathbb{C}^n$ about $\underline{0}$, is the identity on Ω_0. Prove that there are local coordinates at $\underline{0}$ (in general not linear functions of $z_1, \ldots, z_n$) in terms of which F is a linear operator. [Hint: These will be n holomorphic functions $w_1, \ldots, w_n$ defined on a subregion of Ω. To construct them, consider the operator $\frac{1}{|k|}\sum_{j=0}^{k-1}(F')^{-1}F$.]

(4) Prove that (i) $\bar{\partial} \circ \bar{\partial} = 0$ and (ii) $\bar{\partial}\left(\frac{\eta(\underline{\bar{w}}-\underline{\bar{z}})}{|\underline{w}-\underline{z}|^{2n}}\right) = 0$.

(5) (i) Let $f \in Hol(\Omega)$, $\Omega \subseteq \mathbb{C}^n$, $n > 1$. If γ is a constant and $S := f^{-1}(\gamma) \neq \emptyset$, then prove that the level set S is not contained in any compact subset of Ω.
(ii) Using part (i), formulate and prove a version of the maximum principle for holomorphic functions of several variables. (This will be weaker than the version we did in Lecture 37 – think about the 1-variable statement that says the max is achieved at the boundary if one has a continuous extension there.)