Problem Set 3 (Solutions)

(1) Show that \( \frac{R^2 - |z|^2}{|Re^{i\theta_0} - z|^2} \) is harmonic (in \( z \)) on \( D_R \). [Hint: what function is it the real part of?]

(2) (i) Solve the Dirichlet problem on \( \overline{D_1} \) for the following functions on \( \partial D_1 \). (Think of \( \theta \in (-\pi, \pi] \).

(a) \( f(\theta) = \begin{cases} 1, & |\theta| \leq \frac{\pi}{2} \\ -1, & \frac{\pi}{2} < |\theta| \leq \pi \end{cases} \)
(b) \( f(\theta) = 1 - \frac{2}{\pi} |\theta| \)
(c) \( f(\theta) = (|\theta| - \pi)^2 - \frac{\pi^2}{3} \).

First express your answer as a series, then try to identify the holomorphic function of which \( u \) is the real part. [Hint: use part I of Lecture 7, on Fourier series.]

(ii) Using your answer to part (i), evaluate

(a) \( 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \)
(b) \( 1 + \frac{1}{6} + \frac{1}{25} + \frac{1}{49} + \cdots \)
(c) \( 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots \)

(3) True or False: If \( f \) is subharmonic on a region \( \Omega \), then so is \( |f| \).

(4) Let \( \Omega \subseteq \mathbb{C} \) be open, and \( \{f_j\} \subset \mathcal{H}(U) \) a sequence of subharmonic functions converging normally to a function \( f: U \to \mathbb{R} \). Is \( f \) necessarily subharmonic?

(5) Let \( F: U \to V \) be holomorphic and 1-to-1, and \( f: V \to \mathbb{R} \) be subharmonic. (Here \( U \) and \( V \) are regions.) Prove that \( f \circ F \) is subharmonic.

(6) Let \( \Omega \subseteq \mathbb{C} \) be a region, and \( K \subset \Omega \) a compact subset. Show that there exists a constant \( M \) (depending only on \( \Omega \) and \( K \)) such that for every positive harmonic function \( u \in \mathcal{H}(\Omega) \) and pair of points \( z_1, z_2 \in K \), we have \( u(z_2) \leq M \cdot u(z_1) \). [Hint: Harnack.]