Problem Set 4

(1) Show $|z|^\alpha$ is subharmonic (on $\mathbb{C}$) for every positive real $\alpha$.

(2) Let $U = D^*_1$ and $f$ be a continuous function on the boundary (unit circle together with the origin). Then the first 2 steps of Perron’s proof go through: that is, if $u$ (function on $D^*_1$) is the pointwise supremum of the functions in $\mathcal{S}$, then $u$ is bounded above and harmonic. Determine $u$ in both of the following cases:

(a) $f \equiv 1$ on the unit circle and $f(0) = 0$;
(b) $f \equiv 0$ on the unit circle and $f(0) = 1$.

If $u$ is not in $\mathcal{S}$, exhibit a sequence of functions converging to it (which is guaranteed by the second part of the proof). [Hint: use (1), though you only need that $|z|^\alpha$ is subharmonic on $D^*_1$.]

(3) Let $U$ be a bounded and simply connected region which has a barrier at each point, and let $\phi$ be a positive (real valued) continuous function on the boundary $\partial U$. Prove that there exists a holomorphic function $f$ on $U$ whose modulus $|f|$ extends to a continuous function on $\overline{U}$ with $|f| = \phi$ on $\partial U$.

(4) Carry out the following outline of proof that a bounded region $U$ with boundary consisting of two $C^1$ Jordan curves (the domain being homeomorphic to an annulus) is biholomorphic to a region of the form \{ $z$ $|$ $1 < |z| < R$ \}.

(a) Prove that there exists $u \in \mathcal{H}(U)$ such that the limit of $u$ at the outer (resp. inner) boundary is 1 (resp. 0).

(b) Choose a fixed loop $\gamma$ going once around the hole counterclockwise. Set $\omega = -u_y dx + u_x dy$. Show $\int_{\gamma} \omega \neq 0$ by supposing otherwise (i.e. $\int_{\gamma} \omega = 0$) and applying the argument principle to $u + i \oint_{\gamma} \omega$.

(c) Pick $\lambda \in \mathbb{R}$ so that $\int_{\gamma} \lambda \omega = 2\pi$. Show that $\exp \{ \lambda u + i \oint_{\gamma} \lambda \omega \}$ is a well-defined, holomorphic, 1-to-1 and onto mapping from $U$ to \{ $z$ $|$ $1 < |z| < e^{\lambda}$ \}.