Problem Set 5

(1) Prove that there does not exist a Green’s function for the region $\mathbb{C}$. (You may take the point $z_0$ to be 0.)

(2) (a) Find the Green’s function $g(z, z_0)$ for the upper half-plane $\mathbb{H}$ (with singularity at $z_0 \in \mathbb{H}$), and show it extends to a neighborhood of $\mathbb{H} \cup \mathbb{R}$. (b) Given a continuous, real-valued function $f$ on $\mathbb{R}$ with $\lim_{x \to +\infty} f(x) = \lim_{x \to -\infty} f(x) < \infty$, show that $u(z_0) := \int_{\mathbb{R}} \frac{\partial g}{\partial y}(x, z_0)f(x)dx$ solves the Dirichlet problem on $\mathbb{H}$ with $f$ as boundary data.

(3) Show that the period matrix for multiply connected regions (defined in my notes, or in Ahlfors) is symmetric.

(4) Let $\Omega$ be the doubly connected region obtained by removing the vertical strips $[ia, ib]$ and $[\mu + ic, \mu + id]$ from $\mathbb{C}$. This is conformally equivalent to an annulus $A(1, \lambda)$ for a unique $\lambda > 1$.
   (a) Determine $\lambda$. (b) Can you relate this to a complex torus $\mathbb{C}/\Lambda$? (Actually there is more than one way to do this.)

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$^1$The meaning of $\frac{\partial g}{\partial y}(x, z_0)$ is that we are taking the vertical partial derivative in the first variable $z$ and evaluating at $z = x + i0$. We are now thinking of $z_0$ as a variable point in $\mathbb{H}$.