Problem set 6

(1) (a) Compute $\Gamma\left(\frac{1}{2} - n\right)$ for $n \in \mathbb{Z}_{>0}$.

(b) Show that $\Gamma\left(\frac{1}{2} - n + it\right) \to 0$ uniformly for $t \in \mathbb{R}$, as the integer $n \to \infty$.

(2) Show that for $x > 0$, $e^{-x} = \frac{1}{2\pi i} \int_{\sigma=\sigma_0} x^{-s} \Gamma(s) ds$ where $s = \sigma + it$, and the integral is taken on a vertical line with fixed real part $\sigma_0 > 0$, and $-\infty < t < \infty$. [Hint: what is the residue of $x^{-s} \Gamma(s)$ at $s = -n$? Also, use problem 1 and the Stirling formula.]

(3) Use summation by parts to prove the two (separate) statements::

(a) Given a Dirichlet series $\sum_{n \geq 1} \frac{a_n}{n^s}$ (here $\{a_n\} \subset \mathbb{C}$ is just some sequence) converging for some $s_0 = \sigma_0 + it_0$, it converges normally on the set $\{s \mid Re(s) > \sigma_0\}$.

(b) Given a sequence $\{a_n\} \subset \mathbb{C}$, and $C, \sigma_1 \in \mathbb{R}_{>0}$ such that $|a_1 + \cdots + a_n| \leq Cn^{\sigma_1}$ ($\forall n$), the Dirichlet series $\sum \frac{a_n}{n^s}$ converges for $Re(s) > \sigma_1$.

(4) Given $\{a_n\} \subset \mathbb{C}$, write $A_n := a_1 + \cdots + a_n$. Suppose that there are $\sigma_1 \in [0, 1)$ and $\rho \in \mathbb{C}$ such that $|A_n - n\rho| \leq Cn^{\sigma_1}$ ($\forall n$).

(a) Show that $f(s) = \sum \frac{a_n}{n^s}$ is defined for $Re(s) > 1$ (easy).

(b) Show that $f$ has a meromorphic continuation to $Re(s) > \sigma_1$, where it is analytic except for a simple pole at $s = 1$ with residue $\rho$. [Hint: consider $f(s) - \rho\zeta(s)$, and use 3(b) above and Lemma 1 of Lecture 11.]