Problem Set 7

(1) Show that we cannot have \( \liminf_{x \to \infty} \frac{\varphi(x)}{x} < 1 \). (See Theorem 3 of Lecture 19 for context and hint.)

(2) Prove that for \( r > s \) non-negative integers, \( \int_0^1 \int_0^1 \frac{x^r y^s}{1-xy} \, dx \, dy \) exists.
while \( \int_0^1 \int_0^1 \frac{x^r y^s}{1-xy} \, dx \, dy = \zeta(2) - \frac{1}{r^2} - \cdots - \frac{1}{r^r} \). [Hint: use the proof of Lemma 1 in Lecture 20. This is easy.]

(3) Prove \( \zeta(2) \) (and hence \( \pi \), since \( \zeta(2) = \frac{\pi^2}{6} \)) is irrational, by following the outline:

(a) Show that \( \int_0^1 \int_0^1 (1-y)^nP_n(x) \, dx \, dy = (-1)^n \int_0^1 \int_0^1 \phi(x,y) \, dx \, dy \), where \( \phi(x,y) = \frac{xy(1-x)(1-y)}{1-xy} \).

(b) Show that on \([0,1]^2\), \( \phi \) is bounded by \( (\frac{\sqrt{5}-1}{2})^5 \).

(c) Using (2), parts (a) and (b), and Lemma 2 of Lecture 20, derive a contradiction from \( \zeta(2) = \frac{P}{Q} \) (\( P, Q \in \mathbb{Z}, Q \neq 0 \)).

(4) Given a Dirichlet series \( \sum_{n \geq 1} \frac{a_n}{n^s} \) (here \( \{a_n\} \subset \mathbb{C} \) is just some sequence) converging for some \( s_0 = \sigma_0 + it_0 \), it converges normally on the set \( \{s \mid Re(s) > \sigma_0 \} \).

(5) Let \( F(s) = \sum_p p^{-s} \), where the sum runs over all primes. Show that \( F(s) = \log \zeta(s) + G(s) \), where log denotes the principal branch of the logarithm, and \( G(s) \) is analytic on the half-plane \( Re(s) > \frac{1}{2} \). Deduce from this that the function \( F(s) \) does not have a meromorphic continuation to the left of the line \( \sigma = 1 \).

(6) Show that, if \( x \) is sufficiently large, then the interval \([2, x]\) contains more primes than the interval \((x, 2x]\).