Problem set 7

(1) Show that we cannot have \( \liminf_{x \to \infty} \frac{x\phi(x)}{x} < 1 \). (See p. 9 of Lecture 12 for context and hint.)

(2) Prove that for \( r > s \) non-negative integers, \( \int_0^1 \int_0^1 \frac{x^r y^s}{1-xy} dx dy \in \mathbb{Z}[\frac{1}{d^2}] \) while \( \int_0^1 \int_0^1 \frac{x^r y^s}{1-xy} dx dy = \zeta(2) - \frac{1}{1^2} - \cdots - \frac{1}{r^2} \). [Hint: use proof of Lemma 1 in Lecture 13. There’s not much to do.]

(3) Prove \( \zeta(2) \) (and hence \( \pi \), since \( \zeta(2) = \frac{\pi^2}{6} \)) is irrational by following the outline:
   (a) Show \( \int_0^1 \int_0^1 (1-y)^n P_n(x) \frac{dx}{1-xy} dy = (-1)^n \int_0^1 \int_0^1 \phi(x,y) \frac{dx}{1-xy} dy \) where \( \phi(x,y) = \frac{xy(1-x)(1-y)}{1-xy} \).
   (b) Show that on \([0,1]^2\), \( \phi \) is bounded by \( \left\{ \frac{\sqrt{5}-1}{2} \right\}^5 \).
   (c) Using (2), parts (a)&(b), and Lemma 2 of Lect. 13, derive a contradiction from \( \zeta(2) = \frac{P}{Q} \) \((P,Q \in \mathbb{Z}, Q \neq 0)\).