Let $K$ be a compact subset of $U$, then $\exists r$ s.t. for all $z \in K$, $D(z, r) \subset U$. Then, fix $r$, let
\[
K \subset K_r = \overline{\left( \bigcup_{z \in K} D(z, r) \right)}
\]

By Montel's Lemma, $\{f_n\}$ is locally bounded. Then $\exists M_{K_r}$ s.t. $|f(z)| \leq K_r$ ($\forall z \in K_r$ & $f \in \{f_n\}$), by Cauchy,
\[
|f'(z)| = \frac{1}{2\pi} \left| \int_{\partial D(z, r)} \frac{f(\eta)}{(\eta - z)^2} d\eta \right| \leq \frac{1}{2\pi} \frac{2\pi r M_{K_r}}{r^2} = \frac{2\pi M_{K_r}}{r} \overset{\text{def}}{=} N_{K_r}
\]

Thus, $\{f'_n\}$ is also locally bounded, which implies it is a normal family.

2. Let $\beta \in D_1$, $f(z) = \frac{z - \beta}{1 - \beta z}$, $\phi(z) = \frac{\beta}{|\beta|} \frac{z}{z + 1}$, and $g(z) = \frac{(1 - |\beta|)^n}{1 + |\beta|} z$. Then $\phi^{-1}(z) = \frac{\beta z + |\beta|}{-\beta z + |\beta|}$, and $f = \phi \circ g \circ \phi^{-1}$. Set $g_1 = g$ and $g_{n+1} = g \circ g_n$. Then $f_1 = \phi \circ g_1 \circ \phi^{-1}$. Now assume that $f_n = \phi \circ g_n \circ \phi^{-1}$ for some $n$. Then
\[
f_{n+1} = f \circ \phi \circ g_n \circ \phi^{-1} = \phi \circ g \circ \phi^{-1} \circ \phi \circ g_n \circ \phi^{-1} = \phi \circ g_{n+1} \circ \phi^{-1},
\]
so $f_n = \phi \circ g_n \circ \phi^{-1}$ for all $n$.

Since
\[
g_n(z) = \frac{(1 + |\beta|)^n z}{(1 - |\beta|)^n},
\]
we have
\[
f_n(z) = \frac{\beta (1 + |\beta|)^n(\beta z + |\beta|) - (1 - |\beta|)^n(-\beta z + |\beta|)}{|\beta| (1 + |\beta|)^n(\beta z + |\beta|) + (1 - |\beta|)^n(-\beta z + |\beta|)},
\]
and since $|\beta| < 1$, this implies $f_n(z) \rightarrow \frac{\beta}{|\beta|}$.

Because $f$ is holomorphic, $f_n$ is holomorphic, and because $f$ is an automorphism of the unit disk, so is $f_n$. Thus, $|f_n(z)| \leq 1$ on $D_1$, so by Montel's lemma, $f_n$ converges normally to $\frac{\beta}{|\beta|}$ on $D_1$.

3. Let $\Omega \subseteq \mathbb{C}$ be a simply connected region that is symmetric with respect to the real axis. Let $f: \Omega \rightarrow \mathbb{C}$ be a conformal isomorphism onto $D_1$ sending $p \in \Omega$ to $0$, $p \in \mathbb{R}$, and $f'(p) \in \mathbb{R}_+$. Set $g(z) = f(\overline{z})$. Since $\Omega$ is symmetric about the real axis, $\overline{z} \in \Omega$, so the domain of $g$ is $\Omega$. Similarly, $D_1$ is symmetric about the real axis, i.e., $\overline{\alpha} \in D_1$ for all $\alpha \in D_1$, so $g$ is a map from $\overline{\Omega}$ to $D_1$. In fact, $g$ is a biholomorphism onto $D_1$.

First, to show that $g$ is injective, assume that $g(z) = g(w)$. Then $\overline{f(z)} = \overline{f(w)}$, which implies $f(\overline{z}) = f(\overline{w})$, and since $f$ is injective, $\overline{z} = \overline{w}$, hence $z = w$. To show that $g$ is onto, let $w \in D_1$. Then $\overline{w} \in D_1$, and since $f$ is onto, there exists a $t \in \Omega$ such that $f(t) = \overline{w}$. Thus, $f(t) = w$, and since $\Omega$ is symmetric about the real axis, $z := \overline{t} \in \Omega$, and we have
\[
g(z) = f(\overline{z}) = f(t) = w.
\]
Finally, $g$ is holomorphic since $g(z) = f(\overline{z})$ and $f$ is holomorphic. Thus, $g$ is a biholomorphism onto $D_1$.

Moreover, since $p \in \mathbb{R}$,
\[
g(p) = f(\overline{p}) = f(p) = 0 = 0,
\]
and since $f'(p) \in \mathbb{R}_+$,
\[
g'(p) = f'(\overline{p}) = f'(p) = f'(p) \in \mathbb{R}_+.
\]
By the uniqueness part of the Riemann mapping theorem, $f$ and $g$ must be equal, i.e.
Problem 4. Let \( a \in \Omega \). \( \exists \varepsilon > 0 \) such that \( \overline{B(a, \varepsilon)} \subseteq \Omega \). The Mean value Theorem implies that, \( \forall z \in \overline{B(a, \varepsilon/2)} \) and \( \varepsilon/2 \leq r \leq \varepsilon \),
\[
|f_\alpha^2(z)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f_\alpha^2(z + re^{i\theta})| \, d\theta
\]
It follows that,
\[
\int_{\varepsilon/2}^\varepsilon |f_\alpha(z)|^2 \, r \, dr \leq \frac{1}{2\pi} \int_{\varepsilon/2}^\varepsilon \int_0^{2\pi} |f_\alpha(z + re^{i\theta})|^2 \, r \, dr \, d\theta
\leq \frac{1}{2\pi} \int_0^{2\pi} \int_0^r |f_\alpha(z + re^{i\theta})|^2 \, r \, dr \, d\theta
\leq \int_U |f_\alpha(z)|^2 \, dx \, dy
< C
\]
In other words, \( \forall \alpha \) and \( \forall z \in \overline{B(a, \varepsilon/2)} \),
\[
|f_\alpha(z)|^2 < \frac{8C}{3\varepsilon^2}
\]
Hence, \( \{f_\alpha\} \) is locally uniformly convergent. \( \square \)

Let \( g_\alpha = e^{-f_\alpha} \) and \( \mathcal{E} = \{g_\alpha\} \), then
\[
|g_\alpha(z)| = e^{-Re(f_\alpha)} |e^{-Im(f_\alpha)}| \leq 1.
\]
Thus, \( \mathcal{E} \) is locally bounded. And by Montel's lemma, \( \mathcal{E} \) is normal.
For any sequence \( \{f_n\} \subset \mathcal{F} \), sequence \( \{g_n\} \subset \mathcal{E} \) contains a subsequence converging uniformly to some function \( g \). Since \( \{g_n\} \) nowhere zero on \( U \), by Hurwitz:
EITHER
\( g \) is nowhere zero on \( U \). Then, \( \{f_n\} \) converging normally to some function.
OR
\( g \) is identically zero on \( U \). Then, \( \{f_n\} \) tending uniformly to \( \infty \).