1. \[ P(z) = z - \frac{z^{n+1}}{n+1} \]

\[ P'(z) = 1 - z^n \quad \text{roots} \ z_j = e^{\frac{2\pi j}{n}} \]

\[ w_j = P(z_j) = z_j(1 - \frac{1}{n+1}) = \frac{n}{n+1} z_j \]

2. Examine the proof of Bloch's theorem to prove that \( L \geq \frac{1}{2^a} \).

**Proof:** Let \( K(r) = \max \{ |f'(z)| : |z| = r \} \) and let \( h(r) = (1 - r)K(r) \). Then \( h : [0,1] \to \mathbb{R} \) is continuous, \( h(0) = 1 \) and \( h(1) = 0 \). Let \( r_0 = \sup \{ r : h(r) = 1 \} \). Then \( h(r_0) = 1, r_0 < 1 \), and \( h(r) < 1 \) if \( r > r_0 \). Let \( \alpha \) be such that \( |\alpha| = r_0 \) and \( |f'(\alpha)| = K(r_0) \). Then

\[ |f'(\alpha)| = \frac{1}{1 - r_0} \]

Now, if \( |z - \alpha| < \frac{1}{2}(1 - r_0) = \rho_0 \), \( |z| < \frac{1}{2}(1 + r_0) \). Since \( r_0 < \frac{1}{2}(1 + r_0) \), the definition of \( r_0 \) gives

\[ |f'(z)| \leq K\left( \frac{1}{2}(1 + r_0) \right) \]

\[ = \frac{h\left( \frac{1}{2}(1 + r_0) \right)}{1 - \frac{1}{2}(1 + r_0)} \]

\[ < \frac{1}{1 - \frac{1}{2}(1 + r_0)} \]

\[ = \frac{1}{\rho_0} \]

for \( |z - \alpha| < \rho_0 \). Therefore, we get

\[ |f'(z) - f'(\alpha)| \leq |f'(z)| + |f'(\alpha)| < \frac{3}{2\rho_0} \]

According to Schwarz's Lemma, this implies that

\[ |f'(z) - f'(\alpha)| < \frac{3|z - \alpha|}{2\rho_0^2} \]

for \( z \in B(\alpha, \rho_0) = S \). It remains to show that \( f(S) \) contains a disk of radius \( \frac{1}{2\rho_0} \). Define \( g : B(0, \rho_0) \to \mathbb{C} \) by \( g(z) = f(z + \alpha) - f(\alpha) \). Then \( g(0) = 0 \) and \( |g'(0)| = |f'(\alpha)| = \frac{1}{2\rho_0^2} \). If \( z \in B(0, \rho) \), then the line segment \( \gamma = [\alpha, z + \alpha] \) lines in \( S \). By the above we get
\[ |g(z)| = \left| \int_{\gamma} f'(w)dw \right| \leq \frac{|z|}{r_0} < 1. \]

By Corollary 1 in Lecture 19 we have

\[ B(0, \sigma) \subseteq g(B(0, r_0)), \]

where

\[ \sigma = \frac{h_0^2 \left( \frac{1}{2\pi} \right)^2}{6} = \frac{1}{24}. \]

which translates to \( B(f(a), \frac{1}{24}) \subseteq f(S) \). Finally, this implies that \( L \geq \frac{1}{24}. \)

(3) Given \( f \in \mathcal{S} \), write \( f(z) = z + a_2(f) z^2 + a_3(f) z^3 + \cdots \).

In this way we view the \( a_n \) as functions on \( \mathcal{S} \):

\[ a_n : \mathcal{S} \to \mathbb{C}, \]

\[ f \mapsto a_n(f). \]

Let \( f_k \in \mathcal{S} \) be a sequence which is convergent in the

framed topology, with limit \( f \in \mathcal{S} \). To show \( a_n \)

is continuous, we must show \( \lim_{k \to \infty} a_n(f_k) = a_n(f) \).

But \( f_k \) is uniformly convergent on compact subsets of \( \mathcal{D} \),

and (say) \( \mathbb{D}_{D_{V_2}} \) is a compact subset, so

\[ \lim_{k \to \infty} a_n(f_k) = \lim_{k \to \infty} \frac{1}{2\pi i} \int_{\partial D_{V_2}} \frac{f_k(z)}{z^{n+1}} \, dz. \]

\[ = \frac{1}{2\pi i} \int_{\partial D_{V_2}} \lim_{k \to \infty} \frac{f_k(z)}{z^{n+1}} \, dz \]

uniformly converges

\[ = \frac{1}{2\pi i} \int_{\partial D_{V_2}} \frac{f(z)}{z^{n+1}} \, dz = a_n(f), \quad \text{done}. \]
4. Let $\Omega \subseteq \mathbb{C}$ be a simply connected domain, $\alpha \in \Omega$ a point. Let $F: \Omega \rightarrow D$ be the conformal map from $\Omega$ to $D$ with $f(\alpha) = 0$ generated by the Riemann mapping theorem. Consider the function $g(z) := F'(\alpha)(F^{-1}(z) - \alpha)$. Then $g$ is injective, $g(0) = F'(\alpha)(\alpha - \alpha) = 0$, and
\[
g'(0) = F'(\alpha)(F^{-1})'(0) = \frac{F'(\alpha)}{F'(F^{-1}(0))} = \frac{F'(\alpha)}{F'(\alpha)} = 1,
\]
i.e. $g$ is a schlicht function. By the Kobe $\frac{1}{4}$ theorem, $D_{\frac{1}{4}} \subset g(D)$, which implies that
\[
D_{\frac{1}{4}} \subset |F'(\alpha)| (F^{-1}(D) - \alpha) = |F'(\alpha)| (\Omega - \alpha).
\]
Thus, $|z| \leq |F'(\alpha)| |d(\alpha, \partial\Omega)|$ for all $z \in \overline{D_{\frac{1}{4}}}$. Setting $z = \frac{1}{4}$ gives
\[
\frac{1}{4d(\alpha, \partial\Omega)} \leq |F'(\alpha)|.
\]
Now consider the map $h(z) := F(d(\alpha, \partial\Omega)z + \alpha)$. Since $D(\alpha, d(\alpha, \partial\Omega)) \subset \Omega$, we have $h(D) \subset F(\Omega) = D$. Moreover, $h \in \text{Hol}(D)$ because $F \in \text{Hol}(\Omega)$ and $h(0) = F(\alpha) = 0$. Thus, the Schwarz lemma yields
\[
|h'(0)| = |d(\alpha, \partial\Omega)F'(\alpha)| \leq 1,
\]
so we have
\[
\frac{1}{4d(\alpha, \partial\Omega)} \leq |F'(\alpha)| \leq \frac{1}{d(\alpha, \partial\Omega)},
\]
as desired. \qed
(5) Write \( \mathcal{R} = \mathbb{C} \backslash \{0, \infty\} \). \( \text{RMT} \Rightarrow \exists f: \mathcal{R} \xrightarrow{\text{isom}} \mathbb{D}^* \)

sending \( \{0\} \) to \( \{0\} \), hence \( f \) restricts to \( \mathbb{C} \backslash \{1, \infty\} \xrightarrow{\text{isom}} \mathbb{D}^* \).

Were the universal cover \( \mathbb{C} \) or \( \mathbb{P}^1 \), this would result in a map from one of them to \( \mathbb{D}^* \), i.e., a bounded (hole) entire function \( \Rightarrow \) constant \( \times \).

(6) Nontrivial direction is:

\[
\mathbf{h}_{\Gamma_1} = \mathbf{h}_{\Gamma_2} \Rightarrow \Gamma_1, \Gamma_2 \leq \text{PGL}(2, \mathbb{R})
\]

(Note that \( \Gamma_1 \) are in \( \text{PGL}(2, \mathbb{R}) \) but \( \text{PGL}(2, \mathbb{R}) \cong \text{Aut}(\mathbb{H}) \).

**Proof:** Given an isom. \( f: \mathbf{h}_{\Gamma_1} \to \mathbf{h}_{\Gamma_2} \), have

\[
\begin{array}{ccc}
\Gamma_1 & \xrightarrow{f} & \Gamma_2 \\
\mathbf{h}_{\Gamma_1} & \xrightarrow{\text{res}} & \mathbf{h}_{\Gamma_2}
\end{array}
\]

in which \( \mathbf{h} \in \text{Aut}(\mathbb{H}) \).

Looking at this picture differently,

\[
\begin{array}{ccc}
\mathbf{h} & \xrightarrow{f} & \mathbf{h} \\
\mathbf{h}_{\Gamma_1} & \xrightarrow{\text{res}} & \mathbf{h}_{\Gamma_2}
\end{array}
\]

\[
\text{By RMT, } h \text{ clearly acts.}
\]

Cannot have branching like bottom composite does not. So it not 1-1;

is quotient by some group and that would show up as \( \pi_1(\text{image}) \).

But \( \mathbf{h} \) is the image and \( \pi_1(\mathbb{H}) = \{1\} \).
it says that "h identifies quotients by \( \Gamma_2 \) ε \( \Gamma_2 \). If \( \gamma \in \Gamma_1 \) sends \( p \) to \( q \), then there is \( \bar{\gamma} \in \Gamma_2 \) sending \( h(p) \) to \( h(q) \); more precisely, \( \text{hol} h^{-1} = \bar{\gamma} \) belongs to \( \Gamma_2 \). In this way one sees that \( h \Gamma_1 h^{-1} = \Gamma_2 \) inside \( \text{Aut}(H) \leq \text{PGL}_2(\mathbb{R}) \).}

(c) Let \( f \in S \) and \( f(D) \) starlike, \( r \in (0,1) \).

(c) The \( f(D) \) is starlike is expressed by well-definedness of \( \gamma_t(z) := f^{-1}(r f(z))^t \), for any \( t \in (0,1) \).

Clearly

\( \# \) \quad \gamma_0(z) = z, \quad \gamma_1(D) \subset D. \)

We want that \( f(D_r) \) is starlike, i.e. that for any \( t \in (0,1) \)

\( \# \# \) \quad \gamma_t(D_r) \subset D_r.

But \( (\#) \) + Schwarz \( \Rightarrow |\gamma_t(z)| \leq |z| \), from which \( (\# \#) \) is immediate.

It is now clear that \( \frac{d}{d\theta} \log f(re^{i\theta}) \geq 0 \), since otherwise the boundary of the starlike region \( f(D_r) \) would "backtrack along itself" (or \( f \) wouldn't be 1-1), \( \nabla \).

(b) For \( |z| = r \), \( \frac{d}{d\theta} \log f = \frac{\text{Im } \log f(re^{i\theta})}{r} \) \( \text{Im } \log f(re^{i\theta}) = \text{Im } \frac{r e^{i\theta} f'(re^{i\theta})}{f(re^{i\theta})} = \text{Re } \frac{z f'(z)}{f(z)} \).
Set \( g(z) = \int_0^z f(t) \, dt \in \mathcal{K}(D). \) Clearly \( g(0) = 0 \)

\[
\begin{align*}
(\text{a}) \quad g'(z) &= f(z) \quad \therefore \quad g''(z) = \frac{zf'(z) - f(z)}{z^2} = \frac{f'(z) - f(z)}{z^2} \\
&= \text{Re} \left( 1 + \frac{zf''}{g'} \right) = \text{Re} \left( 1 + \frac{z}{f' / f} \cdot \left( \frac{f(z) - f(0)}{z^2} \right) \right) = \text{Re} \left( 1 + \frac{zf'}{f} \right)
\end{align*}
\]

\[= \text{Re} \left( \frac{zf'}{f} \right) \geq 0\]

From parts (a) and (b) of (7).

(b) \[\text{Set } |\zeta| = r, \quad \text{max } \frac{\frac{d}{dt} g(t)}{g(t)} = \frac{1}{\theta} \quad \therefore \quad g(\zeta) = \frac{2\zeta g'(\zeta)}{2}\]

\[\Rightarrow \quad \frac{1}{\theta} \arg \frac{d}{dt} g(r e^{i\theta}) = \frac{1}{\theta} \text{ Im } \log \left( \frac{i \zeta g'(\zeta)}{\zeta^2 + \left( \frac{r e^{i\theta}}{g'(\zeta)} \right)^2} \right) \]

\[= \text{ Im } \left\{ \frac{i \zeta e^{i\theta} + i \zeta g'(\zeta)}{g'(\zeta)} \right\} \]

\[\Rightarrow \quad \text{Re } \left( 1 + \frac{zf''}{g'} \right) \geq 0\]

\[\therefore \text{ Tangent vector to } g(D) \text{ lands only in } \text{counter-clockwise direction as we move counter-clockwise.}\]

\[\Rightarrow \quad g(D_r) \text{ convex } (\text{for each } r \in (0, 1))\]

\[\Rightarrow \quad g(D) \text{ convex.}\]

(c) \[g(z) = z + \sum_{n=2}^{\infty} b_n z^n. \quad \text{Piece in } \mathbb{N}.\]

Set \( G(z) = \sum_{j=1}^m g(s_m^j z^j) \); since any "average" of pts. in a convex set is in the set, we conclude \( G(D) \subset g(D) \).

Further \[ G(z) = b_m z^m + O(|z|^{m+1}), \quad (\exists s_m^j = 0) \]

\[h(z) = \frac{1}{g'(z)} \text{ is defined, sends } D \to D.\]
and so \((g \circ h)(z) = t(z)\)

\[
\Rightarrow h(z) + \sum_{n \geq 2} b_n (h(z))^n = b_m z^m
\]

\[
\Rightarrow h(z) = b_m z^m + h.o.t.
\]

\[
\Rightarrow b_m = \oint_{|z|=r} \frac{h(z)}{z^{m+1}} \, dz
\]

\[
\text{for } r \in (0,1)
\]

\[
\Rightarrow |b_m| \leq \frac{1}{r^m} \quad \Rightarrow \quad |b_m| \leq 1. \quad \text{for any } m \geq 2
\]

\[
\text{with } f(z) = z + \sum_{n \geq 2} a_n z^n
\]

\[
\frac{f(z)}{z} \left(1 + \sum_{n \geq 2} a_n z^{n-1}\right) = g(z) \geq 1 + \sum_{n \geq 2} nb_n z^{n-1}
\]

\[
\Rightarrow a_n = nb_n
\]

\[
\Rightarrow |a_n| \leq n |b_n| \leq n.
\]