Problem Set #8 (Solutions)

1) Write \( Q(z) := \sum_{\lambda \neq 0} \left( \frac{1}{(z - \lambda)^2} - \frac{1}{\lambda^2} \right) \), (note to show this has meaning)

and let \( K \subset \mathbb{C} \) be compact. We have

\[ T_r \subset \{ |u| \leq K \} \cap \{ |u-z| \geq r \} \]

for some \( K, r > 0 \). Clearly the finite part \( \sum_{\lambda \neq 0} \frac{1}{\lambda^2} \) of

the sum defines a holomorphic function on \( \mathbb{C} \) for any open \( \Omega \subset K \). The terms of \( \sum_{\lambda \in A} \) are bounded by

\[ \left| \frac{1}{(z - \lambda)^2} - \frac{1}{\lambda^2} \right| = \left| \frac{|u|}{|z - \lambda|^2 |\lambda|^2} - \frac{|u|}{|z|^2 |\lambda|^2} \right| < \frac{4K}{|\lambda|^2} \]

so the

series is dominated by

\[ \sum_{|\lambda| > 2K} \frac{1}{|\lambda|^2} < C \cdot \int_{|\lambda| > 2K} \frac{d\lambda}{\lambda^2} = \frac{\pi}{2K} \]

(by Residue Theorem) the series converges absolutely and uniformly on \( K \).
2. Prove the following identities:

(a) \( p(z) - p(u) = -\frac{\sigma'(z-u)\sigma(z+u)}{\sigma(z)^2\sigma(u)^2} \)

(b) \( \frac{\zeta'(z)}{p(z)-p(u)} = \zeta(z-u) + \zeta(z+u) - 2\zeta(z) \)

(c) \( \zeta(z+u) = \zeta(z) + \zeta(u) + \frac{1}{2} \frac{p'(z)-p'(u)}{p(z)-p(u)} \)

(d) \( \varphi(z+u) = -\varphi(z) - \varphi(u) + \frac{1}{4} \left( \frac{\varphi'(z)-\varphi'(u)}{\varphi(z)-\varphi(u)} \right)^2 \)

(e) \( \psi(2z) = \frac{1}{4} \left( \frac{\psi'(z)}{\psi(z)} \right)^2 - 2\psi(z). \)

**Proof:** (a) Define \( f(z) = -\frac{\sigma(z-u)\sigma(z+u)}{\sigma(z)^2\sigma(u)^2} \). To see that \( f(z) \) is periodic, it suffices to check that \( f(z+\omega_i) = f(z) \) for \( i = 1, 2 \). Checking this we get

\[
f(z + \omega_i) = -\frac{\sigma((z + \omega_i) + u)\sigma((z + \omega_i) + u)}{\sigma(z + \omega_i)^2\sigma(u)^2} \]
\[
= -\frac{\sigma(z + u)e^{\frac{\varphi(z+u)}{2}}}{\sigma(z)^2\sigma(u)^2} \]
\[
= -\frac{(\sigma(z) - u)e^\frac{\varphi(z)}{2}}{\sigma(z)^2\sigma(u)^2} \]
\[
= f(z). \]

So \( f(z) \) is doubly periodic. Clearly \( p(z) - p(u) \) is doubly periodic. Moreover, \( f(z) \) and \( \varphi(z) - \varphi(u) \) have the same poles and zeroes. So \( \varphi(z) - \varphi(u) = C f(z) \) for some constant \( C \). But now this constant is easily seen to be 1 comparing Laurent series of both sides. So we conclude that \( \varphi(z) - \varphi(u) = -\frac{\sigma(z-u)\sigma(z+u)}{\sigma(z)^2\sigma(u)^2} \).

(b) Holding \( u \) constant, we can take the logarithmic derivative of both sides of (a) to get

\[
\frac{p'(z)}{p(z) - p(u)} = \frac{\sigma'(z-u) + \sigma'(z+u)}{\sigma(z)\sigma(u)} - \frac{2\sigma'(z)\sigma(z)}{\sigma(z)^2\sigma(u)^2} - \frac{2\sigma'(u)\sigma(u)}{\sigma(u)^2} \]
\[
= -\zeta(z-u) + \zeta(z+u) - 2\zeta(z) \]
\[
= \zeta(z-u) + \zeta(z+u) - 2\zeta(z). \]

(c) By holding \( z \) constant in (b) and letting \( u \) vary, we get a similar formula

\[-\frac{p'(u)}{p(z) - p(u)} = -\zeta(z-u) + \zeta(z+u) - 2\zeta(u). \]
Adding this with the result from (b) we get
\[
\frac{\varphi'(z) - \varphi'(u)}{\varphi(z) - \varphi(u)} = 2\zeta(z + u) - 2\zeta(z) - 2\zeta(u).
\]

Finally, solving for \(\zeta(z + u)\), we get the desired result \(\zeta(z + u) = \zeta(z) + \zeta(u) + \frac{1}{2} \frac{\zeta'(z) - \zeta'(u)}{\varphi(z) - \varphi(u)}\).

(d) By differentiating (c) with respect to \(z\) and \(u\), we get the following
\[
\zeta'(z + u) = \zeta'(z) + \frac{1}{2} \left( \frac{\varphi''(z)(\varphi(z) - \varphi(u)) - \varphi'(z)(\varphi'(z) - \varphi'(u))}{(\varphi(z) - \varphi(u))^2} \right).
\]
and
\[
\zeta'(z + u) = \zeta'(u) - \frac{1}{2} \left( \frac{\varphi''(u)(\varphi(z) - \varphi(u)) - \varphi'(u)(\varphi'(z) - \varphi'(u))}{(\varphi(z) - \varphi(u))^2} \right).
\]

Adding these two equations, we arrive at
\[
2\zeta'(z + u) = \zeta'(z) + \zeta'(u) + \frac{1}{2} \left( \frac{(\varphi''(z) - \varphi''(u))(\varphi(z) - \varphi(u)) - (\varphi'(z) - \varphi'(u))(\varphi'(z) - \varphi'(u))}{(\varphi(z) - \varphi(u))^2} \right).
\]

Now, using the differential equation \(|\varphi'(z)|^2 = 4\varphi(z)^3 - 2\varphi(z) - g_1\), we can differentiate again to get \(2\varphi''(z)\varphi'(z) = 12\varphi(z)^2\varphi'(z) - 2g_1\varphi'(z)\), or \(\varphi''(z) = 6\varphi(z)^2 - \frac{g_1}{2}\). So \(\varphi''(z) - \varphi''(u) = 6(\varphi(z)^2 - \varphi(u)^2)\), and the above becomes
\[
\zeta'(z) + \zeta'(u) + \frac{3}{2}(\varphi(z) - \varphi(u))^2.
\]

But now, \(\varphi(z) = -\zeta'(z)\). So this says
\[
-2\zeta'(z + u) = -\zeta'(z) - \zeta'(u) + 3(\varphi(z) - \varphi(u))^2 = 2\varphi(z) + 2\varphi(u) - \frac{1}{2} \left( \frac{\varphi'(z) - \varphi'(u)}{\varphi(z) - \varphi(u)} \right)^2.
\]

Dividing through by \(-2\) gives the desired result.

(e) Using part (d) we get
\[
\varphi(2z) = \lim_{n \to \infty} \varphi(z + u)
\]
\[
= \lim_{n \to \infty} \left[ -\varphi(z) - \varphi(u) + \frac{1}{4} \left( \frac{\varphi'(z) - \varphi'(u)}{\varphi(z) - \varphi(u)} \right)^2 \right]
\]
\[
= -\frac{1}{4} \lim_{n \to \infty} \left( \frac{\varphi'(z) - \varphi'(u)}{\varphi(z) - \varphi(u)} \right)^2 - 2\varphi(z)
\]
\[
= \frac{1}{4} \lim_{n \to 0} \left( \frac{\varphi'(z) - \varphi'(z + h)}{h} \right)^2 \left( \frac{h}{\varphi(z) - \varphi(z + h)} \right)^2 - 2\varphi(z)
\]
\[
= \frac{1}{4} \left( \frac{\varphi'(z)}{\varphi'(z)} \right)^2 - 2\varphi(z).
\]
(a) \[ y^2 = 4x^3 + 4x \]

\[ (y') Q'(u) = 2\sqrt{Q(u)^3 + Q(u)} \]

\[ a = \frac{dy}{dx} = \frac{4(3x^2+1)}{2y} \]

\[ 2y \, dy = 4(3x^2+1) \, dx \]

\[ 4x^3 - 4x - (ax+b)^2 = 4(x - Q(u))^2 \left( x - Q(2u) \right) \]

\[ 4x^3 - a^2x^2 - \ldots = 4x^3 - 4(2Q(u) + Q(2u))x^2 - \ldots \]

\[ \frac{a^2}{4} = 2Q(u) + Q(2u) \]

So \[ Q(2u) = \frac{a^2}{4} - 2Q(u) \]

\[ = \frac{(3Q(u)^2+1)^2}{4Q(u)^3 + Q(u)} - 2Q(u) \]

\[ = \frac{(2Q(u)^2 - 1)^2}{4Q(u)^3 + Q(u)^2 + 1} \]

(b) To double \( (1, 2\sqrt{2}) \), write \( Q(u) = 1 \) and

\[ P(2u) = \frac{(1^2 - 1)^2}{4.1(1^2+1)} = 0 \]

using the equation we determine \( y' \) \[ \text{see } 2(1, 2\sqrt{2}) = (0, 0) \] is a 2-torsion point as the y-coord = 0. So \( (1, 2\sqrt{2}) \) is 9-torsion.
(c) \( P_0 = (x_0, y_0) \)
\[
\mathbf{P}_{i+1} = (x_{i+1}, y_{i+1}) = \frac{2 - (x_i, y_i)}{P_i}
\]
\[
X_{i+1} = \frac{(x_i^2 - 1)^2}{4x_i^2(x_i^2 + 1)} \quad \text{from part (a)}
\]

Claim: suppose \( x_0 = \frac{P_0}{2^{a_0} q_0} \), with \( P_0, q_0 \) odd integers and \( a_0 \) odd > 0.

Then all \( x_i = \frac{P_i}{2^{a_i} q_i} \) are of this form.

\textbf{Pf.:} the calculation is
\[
X_{i+1} = \left( \frac{P_i^2}{4q_i^2} - 1 \right) \frac{P_i}{2^{a_i} q_i} \frac{P_i}{4q_i^2 \left( \frac{P_i^2}{4q_i^2} + 1 \right)}
\]

\[
= \frac{(P_i^2 - 4q_i^2)^2}{2^{a_i+2} q_i (P_i^2 + 4q_i^2)}
\]

Note that \( a_{i+1} = a_i + 2 \).
Now suppose, for some \( N \in \mathbb{N} \), that \( P_0 \) is an \( N \)-torsion point. Then all the \( P_i \) are \( M \)-torsion points for \( M \mid N \); there are finitely many of these, and so \( P_k = P_0 \) for some \( k > 0 \). The problem is that none of the \( x_i \)'s can be equal, by (\( \ast \))!!

So none of the \( P_i \)'s are equal and this is impossible.

4. (a) Let \( \tau = \frac{w_2}{w_1} \), then

\[
\Theta(z+1) = \sum_{n \in \mathbb{Z}} \exp \left\{ \pi i \left( n^2 z + 2n z + 2n \right) \right\} = \Theta(z)
\]

(b) \( W_1 = 1, \ W_2 = \tau \)

\[
\Theta(z) = \sum_{n \in \mathbb{Z}} \exp \left\{ \pi i \left( n^2 z + 2n z \right) \right\}
\]

is an entire function with simple zero at \( \frac{\tau}{2} \).

Checking \( \lim_{\tau \to \frac{\tau}{2}} \frac{\log \Theta(z)}{\log \tau} \), gives us genus = 2.

\( \Rightarrow \) Canonical form becomes

\[
\Theta(z) = z^{0} \prod_{m,n} \left( 1 - \left( \frac{m+n}{2} \right)^2 \right) \ e^{\frac{2z}{m+n} + \frac{1}{2} \left( \frac{2z}{m+n} \right)^2} \ e^{\frac{z}{2} \left( \frac{m+n}{2} \right) \ e^{h(z)}}
\]

\[
= C \cdot \sum_{\text{we } A} \left( z + \frac{\tau + i}{2} \right) e^{h(z)}
\]