I.C. Posets

I.C.1. Definition. A partial order on a set $S$ is a relation $\leq$ such that

\[
\begin{align*}
&x \leq x \\
&x \leq y \text{ and } y \leq z \implies x \leq z \\
&x \leq y \text{ and } y \leq x \implies x = y
\end{align*}
\]

for all $x, y, z \in S$. The pair $(S, \leq)$ is called a poset.

An easy example is $(\mathcal{P}(S), \subset)$.

I.C.2. Definitions. Let $(S, \leq)$ be a poset.

(i) $(S, \leq)$ is totally ordered $\iff x \leq y$ or $y \leq x$ ($\forall x, y \in S$).

(ii) A chain is a subset $C \subset S$ such that $(C, \leq)$ is totally ordered.

(iii) An upper bound\(^6\) for a subset $S' \subset S$ is $x \in S$ such that

\[y \in S' \implies y \leq x.\]

(iv) A maximal element\(^7\) of $S$ is $x \in S$ such that

\[x \leq y \text{ and } y \in S \implies x = y.\]

I.C.3. Zorn’s Lemma. If every chain in $S$ has an upper bound, then $S$ has a maximal element.

This is needed for:

- $\exists$ of bases for $\infty$-dimensional vector spaces (i.e. a linearly independent subset contained by no proper linear subspace);
- $\exists$ and $!$ of the algebraic closure of a field;\(^8\)
- $\exists$ of transcendence bases for arbitrary field extensions;
- $\exists$ of maximal (proper) ideals containing a given proper ideal (for rings with uncountably many elements); and
- (in analysis) stuff like the Hahn-Banach extension.

Zorn’s Lemma follows from (indeed, is equivalent to) the

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\(^6\)These need not exist or be unique in general: consider various subsets $S' \subset \mathbb{R}$ of the reals.

\(^7\)This need not satisfy $y \leq x \forall y \in S$, unless of course $S$ is totally ordered.

\(^8\)In mathematics, the symbol “!” stands for “unique” (or uniqueness, or uniquely).
I.C.4. Axiom of Choice. Given a family of nonempty sets \( \{X_i\}_{i \in I} \), there exists a “choice function” \( f \) defined on \( I \) such that \( f(i) \in X_i \) (\( \forall i \)). Alternately, \( \exists f = \{f(i)\}_{i \in I} \in \prod_{i \in I} X_i \) – that is, the Cartesian product is nonempty.

(Clearly, this is only needed when \( I \) is infinite.) People make a fuss about using it because it renders your argument nonconstructive.

Sketch of proof that AOC \( \implies \) ZL. Let \( (S, \leq) \) be a poset in which all chains have an upper bound (write “UB”). For each \( x \in S \), set \( \varphi(x) := \{ y \in S \mid y > x \} \in \mathcal{P}(S) \), and assume no \( x \) is maximal (i.e. no \( \varphi(x) = \emptyset \)). By I.C.4, there exists a choice function \( f \) on \( \varphi(S) \) (a subset of \( \mathcal{P}(S) \)), with \( f(\varphi(x)) \in \varphi(x) \).

Now, fixing \( x \in S \), define a “sequence” in \( S \) by transfinite\(^9\) recursion:

\[
\begin{cases}
   x_0 := x, \\
   x_{\alpha+1} := f(\varphi(x_\alpha)) (> x_\alpha) \quad \text{for any ordinal number } \alpha \\
   x_\alpha := f(\varphi(\text{UB}\{x_\beta \mid \beta < \alpha\})).
\end{cases}
\]

This “goes on forever”, so that \( \alpha \mapsto x_\alpha \) yields an injection \( \text{Ord} \hookrightarrow S \) — which is impossible because \( \text{Ord} \) is not a set. \( \square \)

This was just to give an idea; if you want more than that, pick up Halmos’s “Naive set theory” book.

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\(^9\)There are arguments that avoid transfinite induction, but they take longer to even partially understand. You can think of an ordinal number as an isomorphism class of well-ordered sets (which are totally ordered sets each of whose subsets has a least element). The class \( \text{Ord} \) of ordinal numbers is not a set – it is “too big”.