Problem Set 8

(Hand in all.)

(1) [Jacobson p. 111 #17] Determine the ideals and maximal ideals and prime ideals of \( \mathbb{Z}_{60} \cong \mathbb{Z}/(60) \).

(2) [Jacobson p. 118 #5] Let \( R \) be a commutative ring, and \( S \) a submonoid of the multiplicative monoid of \( R \). In \( R \times S \), define \((a, s) \sim (b, t)\) if there exists a \( u \in S \) such that \( u(at - bs) = 0 \). Show that this is an equivalence relation in \( R \times S \). Denote the equivalence class of \((a, s)\) as \( a/s \) and the quotient set consisting of these classes as \( RS^{-1} \). Show that \( RS^{-1} \) becomes a ring relative to \( a/s + b/t = (at + bs)/st \), \( (a/s)(b/t) = ab/st \), \( 0 = 0/1 \), and \( 1 = 1/1 \). Show that \( a \mapsto a/1 \) is a homomorphism of \( R \) into \( RS^{-1} \), and that this is a monomorphism if and only if no element of \( S \) is a zero-divisor in \( R \). Show that the elements \( s/1, s \in S \), are units in \( RS^{-1} \).

(3) [Jacobson p. 126 #2] Show that \( \sqrt{3} \notin \mathbb{Q}[\sqrt{2}] \) and that the real numbers \( 1, \sqrt{2}, \sqrt{3}, \sqrt{6} \) are linearly independent over \( \mathbb{Q} \). Show that \( u = \sqrt{2} + \sqrt{3} \) is algebraic and determine an ideal \( I \) such that \( \mathbb{Q}[x]/I \cong \mathbb{Q}[u] \).

(4) [Jacobson p. 133 #9] Show that the ideal \((3, x^3 - x^2 + 2x - 1)\) in \( \mathbb{Z}[x] \) is not principal.

(5) [Jacobson p. 133 #13] Prove Wilson’s theorem: if \( p \) is a prime in \( \mathbb{Z} \), then we have \((p-1)! \equiv -1 \) \((p)\).

(6) A commutative ring is local if it has a unique maximal ideal.

(a) Using problem (2) above with \( R = \mathbb{Z}, S = \{b \in \mathbb{Z} \mid p \nmid b\} \), prove that \( RS^{-1} \) is local.

(b) If \( \mathcal{R} \) is a local ring with unique maximal ideal \( m \), prove that \( a \in \mathcal{R} \) is a unit if and only if \( a \notin m \). [Hint: you will need to use (III.D.27).]

(7) Find an inverse (fractional ideal) for \( I = (15 + 25\sqrt{-10}, 25 - 40\sqrt{-10}) \subset \mathbb{Z}[\sqrt{-10}] = R \).

(8) [Jacobson p. 140 #3] (Newton’s identities. See Jacobson. Statement of problem is long but solution need not be.)