II.B. Classical groups (a brief zoology)

So far we have made a reasonable case that to avoid pathologies, we should stick to nondegenerate forms that are either symmetric or alternating. The classical groups that we study in the sections that follow are primarily *the groups* $\mathrm{Aut}_{\mathbb{F}}(V, B)$ *of* $\mathbb{F}$*-linear* ***isometries*** of such bilinear forms, i.e. isomorphisms $T\colon V \to V$ satisfying

$$B(T(u), T(v)) = B(u, v).$$

To delineate the classification a bit, we shall work with matrices with respect to a fixed basis $e = \{e_1, \ldots, e_n\}$ of V, and note that writing arbitrary invertible transformations T in this fashion (as $[T]_e = X$) induces a group isomorphism

$$\text{(II.B.1)} \qquad \mathrm{Aut}_{\mathbb{F}}(V) \cong \mathrm{GL}_n(\mathbb{F}) := \{X \in M_n(\mathbb{F}) \mid X \text{ is invertible}\},$$

where the RHS is called the **general linear group**. It has the **special linear group**

$$\mathrm{SL}_n(\mathbb{F}) := \{X \in M_n(\mathbb{F}) \mid \det X = 1\}$$

as a subgroup.

Now assume B nondegenerate, and set $M := [B]_e$. Then we have $\det M \neq 0$; and B is symmetric resp. alternating exactly when ${}^tM = M$ resp. ${}^tM = -M$. As before, writing isometries with respect to the basis induces isomorphisms

$$\text{(II.B.2)} \qquad \mathrm{Aut}_{\mathbb{F}}(V, B) \cong \left\{X \in \mathrm{GL}_n(\mathbb{F}) \mid {}^tXMX = M\right\}.$$

The groups in (II.B.2) are called **orthogonal** (written $\mathrm{O}(V, B)$ or in the symmetric case and **symplectic** (and written $\mathrm{Sp}(V, B)$) in the alternating case. They are *a priori* subgroups of $\mathrm{GL}_n(\mathbb{F})$, but the symplectic groups are in fact subgroups of $\mathrm{SL}_n(\mathbb{F})$. Since elements of $\mathrm{O}(V, B)$ can have determinant -1, the orthogonal groups are not, but we can define index-2 subgroups called *special orthogonal* groups by $\mathrm{SO}(V, B) := \mathrm{O}(V, B) \cap \mathrm{SL}_n(\mathbb{F})$.

As we shall see later, the symplectic groups only exist for n even, and (fixing n and assuming $\mathrm{char}(\mathbb{F}) \neq 2$) are all isomorphic, justifying the standard notation $\mathrm{Sp}_n(\mathbb{F})$. For orthogonal groups, this "uniqueness" is only true over fields with $\mathbb{F}^* = (\mathbb{F}^*)^2$, like $\mathbb{C}$, in which case we can write $\mathrm{O}_n(\mathbb{F})$ and $\mathrm{SO}_n(\mathbb{F})$. In fact, this completes the classification of the simple complex classical groups: where $\mathbb{1}_n$

Cartan type	A_r	B_r	C_r	D_r
Lie group	$\mathrm{SL}_{r+1}(\mathbb{C})$	$\mathrm{SO}_{2r+1}(\mathbb{C})$	$\mathrm{Sp}_{2r}(\mathbb{C})$	$\mathrm{SO}_{2r}(\mathbb{C})$
$M = [B]_e$	—	$\mathbb{1}_{2r+1}$	$\begin{pmatrix} 0 & \mathbb{1}_r \\ -\mathbb{1}_r & 0 \end{pmatrix}$	$\mathbb{1}_{2r}$

always denotes the $n \times n$ identity matrix.

To get an idea of what happens for $\mathbb{F} = \mathbb{R}$, first consider the orthogonal groups: the symmetric matrices $M = \begin{pmatrix} -\mathbb{1}_p & 0 \\ 0 & \mathbb{1}_q \end{pmatrix}$ are not cogredient for different p and q, and so the resulting real groups of type B_r resp. D_r, namely $\mathrm{SO}(p,q)$ (with $p+q = 2r+1$ resp. $2r$), are not isomorphic for different pairs (p,q).[3] The cases $\mathrm{SO}(p,0) = \mathrm{SO}(p) = \mathrm{SO}(0,p)$ are called *definite* special orthogonal groups, and the other *indefinite*. Actually type D_r has one more real group, the *quaternionic orthogonal group*

$$\mathrm{O}^*(2r) := \{g \in \mathrm{GL}_n(\mathbb{H}) \mid g^*(\mathbf{i}\mathbb{1}_n)g = \mathbf{i}\mathbb{1}_n\},$$

which can also be identified with the *complex* matrices

$$\{X \in \mathrm{SO}_{2n}(\mathbb{C}) \mid J\bar{X} = XJ\}$$

where $J = \begin{pmatrix} 0 & \mathbb{1}_r \\ -\mathbb{1}_r & 0 \end{pmatrix}$.

There is another important series of real Lie groups that appears in type A_n: the **unitary** groups. More generally, for any field $\mathbb{F}$ possessing an involution $\rho\colon \mathbb{F} \to \mathbb{F}$ (i.e. $\rho^2 = \mathrm{Id}_{\mathbb{F}}$), with fixed field $\mathbb{F}_0$, we can define these groups. For simplicity write $V = \mathbb{F}^n$ and "$\overline{(\ )}$" for $\rho(\ )$, and suppose that $M \in M_n(\mathbb{F})$ is invertible with ${}^tM = \bar{M}$.

[3] Here (p,q) and (q,p) are considered to be the "same" as pairs.

Define a *Hermitian form* by $H(v,w) := {}^t v M \bar{w} = \overline{H(w,v)}$, with corresponding unitary group

$$\begin{aligned} U(H) &:= \{Z \in M_n(\mathbb{F}) \mid {}^t Z M \bar{Z} = M\} \\ &= \{Z \mid H(Zv, Zw) = H(v,w) \ (\forall v, w \in V)\}. \end{aligned}$$

These are to be regarded as "linear algebraic groups over $\mathbb{F}_0$" (*not* $\mathbb{F}$) because of the "half-conjugate-linearity" (or "sesquilinearity") of H in $\mathbb{F}$: the groups are cut out by polynomial equations in Z_{ij} *and their conjugates* $\bar{Z}_{ij}$; but writing $\mathbb{F} = \mathbb{F}_0(\alpha)$ and $Z = X + \alpha Y$, these become honest polynomial equations in the "real" and "imaginary" parts $X_{ij}, Y_{ij} \in \mathbb{F}_0$.

If we now take $\mathbb{F} = \mathbb{C}$, $\mathbb{F}_0 = \mathbb{R}$, and $M = \begin{pmatrix} -\mathbb{1}_p & 0 \\ 0 & \mathbb{1}_q \end{pmatrix}$, the resulting group is denoted $U(p,q)$ and is of type A_{p+q-1}, a real form of $\mathrm{SL}_{p+q}(\mathbb{C})$. We can further intersect with $\mathrm{SL}_{p+q}(\mathbb{C})$ to get the (indefinite and definite) special unitary groups. With the addition of quaternionic special linear groups $\mathrm{SU}^*(2m) := \mathrm{SL}_m(\mathbb{H})$ in type A_{2m-1} and certain "quaternionic unitary groups" $\mathrm{Sp}(p,q)$ in type C_{p+q}, this essentially completes the classification over $\mathbb{R}$.

II.B.3. EXAMPLES. (A) As a simple exercise, convince yourself that $\mathrm{Sp}_2(\mathbb{C}) = \mathrm{SL}_2(\mathbb{C})$, and that the "real" groups $\mathrm{SO}(2)$ (rotations in the plane) and $U(1)$ ($= U(0,1)$) (unit circle in $\mathbb{C}^*$) are isomorphic.

(B) To elucidate the remarks on real forms in the introduction, the point is that all the matrix groups just discussed are cut out by polynomial equations in the matrix entries, like $\det X = 1$. For instance, we can exhibit $\mathbb{R}^*$ or $\mathbb{C}^*$ as a 2×2 (real or complex) matrix group in this way by setting $X_{12} = X_{21} = 0$ and $X_{11}X_{22}\,(= \det X) = 1$ (and plugging in real or complex numbers); the isomorphism sends $x \mapsto \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}$ ($x \in \mathbb{R}^*$ or $\mathbb{C}^*$).

(C) We can exhibit the unit circle S^1 as a 2×2 real matrix group by setting $X_{11} = X_{22}$, $X_{12} = -X_{21}$, and $X_{11}^2 + X_{21}^2\,(= \det(X)) = 1$. This time the isomorphism sends a complex number $z = x + \mathbf{i}y$ of absolute value 1 to $\begin{pmatrix} x & -y \\ y & x \end{pmatrix}$. Another easy exercise is to show that the real matrix group just described is a real form of $\mathbb{C}^*$, that is, to give a multiplicative group isomorphism from the nonzero complex numbers to *complex* matrices of the form $\left(\begin{smallmatrix} u & -v \\ v & u \end{smallmatrix}\right)$.

From the last example it should now be clear how we can turn "sesquilinear" algebraic groups over $\mathbb{F} = \mathbb{F}_0(\alpha)$ into honest linear algebraic groups over $\mathbb{F}_0$. We simply replace every Z_{ij} by a 2×2 block $\begin{pmatrix} X_{ij} & \alpha^2 Y_{ij} \\ Y_{ij} & X_{ij} \end{pmatrix}$! Of course, $\alpha = \mathbf{i}$ if $\mathbb{F}_0 = \mathbb{R}$.