III. Representation Theory

Let G be a finite group and $\mathbb{F}$ a field. A **representation** of G over $\mathbb{F}$ is a finite dimensional $\mathbb{F}$-vector space V together with a homomorphism

$$\pi \colon G \to \mathrm{Aut}_{\mathbb{F}}(V).$$

Among the basic classification results one finds that:

- all representations decompose as direct sums of a finite collection of *irreducible representations*;
- these "irreps" are in 1-to-1 correspondence with the conjugacy classes of G;
- they all occur in the "regular representation" of G on the vector space $\mathbb{F}^{|G|} = \mathbb{F}\langle \{e_g\}_{g\in G}\rangle$, where $g.e_{g'} := e_{gg'}$; and
- since each irrep V occurs $\dim V$ times in the regular representation, the sum of the squares of their dimensions is $|G|$.

The approach we take begins from the more general setting of modules over a semisimple ring, briefly discussed in [**Algebra I**, §IV.B]. These are rings whose modules all decompose into direct sums of irreducibles. By a theorem of Artin and Wedderburn, such a ring is a product of matrix algebras over division rings, and contains copies of all its simple modules.

How is this related to representations of groups? The point is that these are the same thing as (finitely generated) $\mathbb{F}[G]$-modules, where $\mathbb{F}[G]$ is the group ring of G. Moreover, the regular representation is just $\mathbb{F}[G]$ as a module over itself. So the above results follow naturally once we can show that $\mathbb{F}[G]$ is a semisimple ring, which will follow from a theorem of Maschke. We will then conclude with a bit of character theory.

III.A. Semisimple modules and rings

Since we are going to be dealing with *non*commutative rings, we should remember that there are left and right modules and ideals. The notation ${}_R R$ [resp. R_R] means that we are considering R as a left- [resp. right-] R-module. We will *usually* deal with left-R-modules.

III.A.1. DEFINITION. Let R be a ring, and M a left R-module.
(i) M is **simple** if $\{0\}$ and M are the only submodules. (If M is ${}_R R$, this is also called a simple left ideal.)
(ii) M is **semisimple** if, for every submodule $N \subseteq M$, there exists a submodule N' such that $M = N \oplus N'$ as an R-module.

Obviously, simple modules are semisimple. Much less obviously:

III.A.2. THEOREM. *The following are equivalent:*
(a) *M is semisimple;*
(b) *M is isomorphic to a direct sum of simple R-modules;*
(c) *M is an internal direct sum of simple R-submodules; and*
(d) *M is a sum of simple R-submodules.*

PROOF. The equivalence of (a), (b) and (c) is [**Algebra I**, IV.B.33]. Clearly (c) implies (d). For (d) $\Longrightarrow$ (c), suppose $M = \sum_{i \in \mathcal{I}} M_i$, with M_i simple. Zorn's lemma conjures a nonempty subset $\mathcal{J} \subset \mathcal{I}$ which is maximal with respect to the property that $\sum_{j \in \mathcal{J}} M_j = \oplus_{j \in \mathcal{J}} M_j$. For any $i \in \mathcal{I}$, $M_i \cap (\oplus_{j \in \mathcal{J}} M_j)$ is (by simplicity of M_i) either M_i or $\{0\}$. Since the latter would contradict maximality of $\mathcal{J}$, we see that $\oplus_{j \in \mathcal{J}} M_j$ contains every M_i hence is all of M. $\square$

III.A.3. PROPOSITION. *If M is semisimple, then any sub- or quotient-module is also semisimple.*

PROOF. Given submodules $N \subset M' \subset M$, there exists N' such that $N \oplus N' = M$; hence $M' = N \oplus (N' \cap M')$ (by writing $m' = n + n'$ $\Longrightarrow$ $n' = m' - n \in M'$).

Given $U \subset M/M'$, by the 1st isomorphism theorem we have $M' \subset \tilde{U} \subset M$ such that $\tilde{U}/M' = U$. By semisimplicity of M, there exists $\tilde{V} \subset M$ with $M = \tilde{U} \oplus \tilde{V}$; and we set $V := (\tilde{V} + M')/M'$.

Certainly $U + V = M/M'$. Moreover, $\tilde{V} \cap \tilde{U} = \{0\}$ together with $\tilde{U} \supset M' \implies (\tilde{V} + M') \cap \tilde{U} = M' \implies U \cap V = \{0\}$. So $M/M' = U \oplus V$. □

As mentioned in §I.J, there is a version of **composition series** for R-modules: this is a finite decreasing filtration

$$M = M_0 \supset \cdots \supset M_{i-1} \supset M_i \supset \cdots \supset M_n = \{0\}$$

of M by submodules, such that the successive quotients M_{i-1}/M_i (called graded pieces) are simple R-modules.

Just as for groups, it may or may not exist, but the difficulties are different: it is not hard to show that existence is equivalent to M being both **Artinian** and **Noetherian**, which is to say that the descending resp. ascending chain conditions — that there are no infinite such chains — both hold. (One direction is clear; for the other you'll need to intersect the terms of a chain with a given composition series.) Viewed as modules over themselves, $\mathbb{Z}$ does *not* have a composition series (why?), while $M_n(\mathbb{C})$ does (see III.A.6 for a hint).

This comes with a version of Jordan-Hölder as well:

III.A.4. THEOREM. *If M has a composition series, then the isomorphism classes and multiplicities of the simple graded pieces* (*though not their location in the series*) *are unique.*

PROOF. This is similar to the proof of I.J.13, but there are a couple of nontrivial differences. The goal is to show that any two CS are equivalent ('$\equiv$') in the sense of the Theorem: same length and same factors (up to order). We induce on n := the minimal length of a CS for M, assuming the result for modules admitting CS of "shorter length". Suppose (a) $M \supset M_1 \supset M_2 \supset \cdots$ and (b) $M \supset N_1 \supset N_2 \supset \cdots$ are CS, where we may assume (a) has length n. If $M_1 = N_1$ we are done by induction.

If $M_1 \neq N_1$, then set $L_2 := M_1 \cap N_1$, take a CS $L_2 \supset L_3 \supset \cdots$ for it (which certainly exists, e.g. by intersecting $L_2 \cap M_i$ and removing repetitions), and consider (c) $M \supset M_1 \supset L_2 \supset L_3 \supset \cdots$ and (d) $M \supset N_1 \supset L_2 \supset L_3 \supset \cdots$. These are CS because $M_1/L_2 =$

$M_1/(M_1 \cap N_1) = (M_1 + N_1)/N_1 = M/N_1$ (the last '=' from simplicity of M/N_1) makes M_1/L_2 simple, and the same goes for N_1/L_2.

Now M_1 has a CS of length $n-1$, so (a) $\equiv$ (c) by induction. In particular, (c) has length n. But then then (d) does too, so N_1 has a CS of length $n-1$ and (d) $\equiv$ (b) by induction. Finally, (c) $\equiv$ (d) by inspection (with 1^{st} and 2^{nd} graded pieces swapped), and so we conclude finally that (a) $\equiv$ (b). $\square$

Turning to rings, we have the

III.A.5. DEFINITION. Let R be a ring.
(i) R is **semisimple** if every left R-module is semisimple.
(ii) R is **simple** if it is semisimple and (0) and R are its only 2-sided ideals.

III.A.6. EXAMPLE. Consider $R = M_n(\mathbb{C})$. Matrices with zeroes in all but the j^{th} column yield a copy of $\mathbb{C}^n$ which is closed under left-multiplication by R. In this way we get n copies of the same left R-module inside R itself, which have ${}_R R$ (i.e. R regarded as left R-module) as their direct sum. (You can also think of them as left ideals.) So ${}_R R$ is semisimple but not simple.

On the other hand, R is simple *as a ring*, because the *2-sided* ideal generated by *any* nonzero element of R is R itself. (Any nonzero matrix has a nonzero entry. Multiply on left and right by $\mathbf{e}_{ii}$ and $\mathbf{e}_{jj}$ to get a matrix with *only* that nonzero entry. Then multiply by $\mathbf{e}_{k\ell}$'s to move this nonzero entry to every spot.)

So R simple does not imply ${}_R R$ simple, but we *do* have

III.A.7. PROPOSITION. *R semisimple $\iff$ ${}_R R$ semisimple.*

PROOF. The forward implication is immediate from III.A.5(i). For the converse, let M be a left R-module, and assume ${}_R R$ is semisimple. Given any cyclic submodule $Rz \subset M$, there is an R-module homomorphism $R \twoheadrightarrow Rz$ with kernel the left ideal $\text{ann}(z)$, and $Rz \cong R/\text{ann}(z)$. By III.A.3, $R/\text{ann}(z)$ is semisimple. So $M = \sum_{z \in M} Rz$ is a sum of simple submodules, hence semisimple by III.A.2. $\square$

III.A.8. COROLLARY. *A semisimple ring is the direct sum of finitely many simple left ideals.*

PROOF. By III.A.7 and III.A.2, ${}_RR$ is a (possibly infinite) direct sum of simple left ideals $\oplus_{j\in\mathcal{J}} I_j$. So $1 = 1_R$ is a *finite* sum $\sum_{j=1}^m \imath_j$, with $\imath_j \in I_j$. But then $R = R1 = \sum_{j=1}^m R\imath_j = \sum_{j=1}^m I_j = \oplus_{j=1}^m I_j$. $\square$

III.A.9. REMARK. Some further notes and warnings are in order:

(a) It would be logical to call R in III.A.5(i) *left* semisimple, but this is unnecessary: it will turn out to be equivalent to the same condition involving right R-modules.

(b) The definition of semisimplicity for a ring R here is stronger than in some texts, which require only that the *Jacobson radical* (the intersection of left annihilators of all simple left R-modules) be zero. To make this equivalent to our definition, you need to also require R to be (left) **Artinian**, i.e. that there is no infinite descending sequence of left ideals.

(c) The same thing as in (b) goes for simplicity of R, in that the condition on ideals (which is often all that is reqiured) does not imply semisimplicity in our sense. For instance, if $\mathbb{C}\langle\langle x, \partial\rangle\rangle$ is the free algebra on 2 generators, then the Weyl algebra $\mathbb{C}\langle\langle x, \partial\rangle\rangle/(\partial x - x\partial - 1)$ has no nontrivial proper 2-sided ideals, but is not Artinian, as $(\partial) \supset (\partial^2) \supset (\partial^3) \supset \cdots$ violates the descending chain condition on left ideals.