(1) Prove that an Artinian commutative domain is a field. [Hint: to find an inverse for $a \neq 0$, consider $(a) \supset (a^2) \supset (a^3) \supset \cdots$.]

(2) Show that every homomorphic image of a left Noetherian [resp. Artinian] ring is left Noetherian [resp. Artinian].

(3) If $S$ is a multiplicative subset of a commutative ring $R$, show that (a) $S^{-1}(\text{Rad } I) = \text{Rad}(S^{-1}I)$ and (b) $S^{-1}R$ is Noetherian if $R$ is Noetherian.

(4) Show that a commutative ring is local if and only if for all $r, s \in R$, $r + s = 1$ implies $r$ or $s$ is a unit.

(5) Let $p$ be a prime in $\mathbb{Z}$; then $(p)$ is a prime ideal. What can be said about the relationship between $\mathbb{Z}_p$ and the localization $\mathbb{Z}(p)$? Describe $\mathbb{Z}(p)$ as a subset of the rational numbers.

(6) Find the nilradical of $\mathbb{Z}_n$ ($n \in \mathbb{N}$).

(7) Prove the five-lemma (Remark IV.B.7(iii)).