Problem Set 10

Some of problems 1-5 use material (on localization and prime ideals) we'll cover on Friday.

(1) Prove that an Artinian commutative domain is a field. [Hint: to find an inverse for $a \neq 0$, consider $(a) \supset (a^2) \supset (a^3) \supset \cdots$.]

(2) Show that every homomorphic image of a left Noetherian [resp. Artinian] ring is left Noetherian [resp. Artinian].

(3) If S is a multiplicative subset of a commutative ring R, show that (a) $S^{-1}(\mathrm{Rad} I) = \mathrm{Rad}(S^{-1}I)$ and (b) $S^{-1}R$ is Noetherian if R is Noetherian.

(4) Show that a commutative ring is local if and only if for all $r, s \in R$, $r + s = 1$ implies r or s is a unit.

(5) Let p be a prime in $\mathbb{Z}$; then (p) is a prime ideal. What can be said about the relationship between $\mathbb{Z}_p$ and the localization $\mathbb{Z}_{(p)}$? Describe $\mathbb{Z}_{(p)}$ as a subset of the rational numbers.

(6) For $G = \mathfrak{S}_3$ (symmetric group), show that $\mathbb{Q}[G] \cong \mathbb{Q} \times \mathbb{Q} \times M_2(\mathbb{Q})$ and compute the central idempotents of $\mathbb{Q}[G]$ which give this decomposition of $\mathbb{Q}[G]$ into its simple components. Compute, similarly, the decompositions of $\mathbb{Q}[G_1]$, $\mathbb{Q}[G_2]$ where G_1 is the Klein 4-group and G_2 the quaternion group.

(7) Show that, over $\mathbb{Q}$, $\mathfrak{A}_5$(alternating group) has four irreducible represntations ("irreps"), of dimensions 1, 4, 5, 6 respectively.

(8) Show that the 3-dimensional irrep $\mathbf{st} \otimes \mathbf{sgn}$ of $\mathfrak{S}_4$ is equivalent (i.e. isomorphic as $\mathbb{C}[\mathfrak{S}_4]$-modules) to the representation of $\mathfrak{S}_4$ as the group of rotational symmetries of the cube (or octahedron). [Hint: compute the character of the latter.]

(9) Suppose the character table of a finite group has the following two rows ($\zeta_3 = e^{\frac{2\pi i}{3}}$)

1	1	1	ζ_3^2	ζ_3	ζ_3^2	ζ_3
2	-2	0	-1	-1	1	1

corresponding to characters of two irreps. (The first column gives the value on $\{1\}$, and the remaining columns the values on the other six conjugacy classes.) Determine the rest of the character table.