Problem Set 10

Do (3)-(8), and your choice of (1) or (2).

1. Let $V$ be a $k[G]$-module and $H$ a subgroup in $G$ of finite index not divisible by $\text{char}(k)$. Modify the proof of Maschke’s theorem to show the following: if $V$ is semisimple as a $k[H]$-module, then $V$ is semisimple as a $k[G]$-module.

2. (a) For $G = \mathfrak{S}_3$ (symmetric group), show that $Q[G] \cong Q \times Q \times M_2(Q)$ and compute the central idempotents of $Q[G]$ which give the decomposition of $Q[G]$ into its simple components. (That is, use (III.D.8) to compute the elements $\sum_{i=1}^{n_i} e_i^k$.) (b) Do the same thing for the Klein 4-group.

3. Let $Q$ be the quaternion group. Find the irreducible representations of $Q$ and product decomposition of $F[Q]$, for $F = \mathbb{R}$ (or $Q$) and $C$.

4. Recall that $\mathfrak{S}_5$ acts transitively on its six Sylow 5-subgroups. This gives a 6-dim’l representation. Compute its character just on the “even conjugacy classes”, which amounts to its character as a representation of $\mathfrak{A}_5$. Now it has an obvious fixed vector (i.e. copy of the trivial representation). Show that the complementary 5-dimensional subrepresentation is irreducible as a representation of $\mathfrak{A}_5$ (hence $\mathfrak{S}_5$) by using its character.

5. Show that, over $Q$, the alternating group $\mathfrak{A}_5$ has four irreducible representations (“irreps”), of dimensions 1, 4, 5, 6 respectively. [Hint: look at how $\mathfrak{A}_5$ appears in the section on Burnside from last term. What is the trace of a rotation by 72 degrees?] (c) Show that the 3-dimensional irrep $\text{st} \otimes \text{sgn}$ of $\mathfrak{S}_4$ is equivalent (i.e. isomorphic as $\mathbb{C} [\mathfrak{S}_4]$ modules) to the representation of $\mathfrak{S}_4$ as the group of rotational symmetries of the cube (or octahedron). [Hint: compute the character of the latter.]

6. Show that the character of a tensor product of representations is the product of the characters. Use this together with III.D.14(i) to compute the decompositions of $\text{st} \otimes W$ and $\text{st} \otimes \text{st}$ into irreps of $\mathfrak{S}_4$.

7. Suppose the character table of a finite group has the following two rows ($\zeta_3 = e^{2\pi i / 3}$)

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>1</th>
<th>1</th>
<th>$\zeta_3$</th>
<th>$\zeta_3^2$</th>
<th>$\zeta_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>-2</td>
<td>0</td>
<td>$-1$</td>
<td>$-1$</td>
<td>1</td>
</tr>
</tbody>
</table>

corresponding to characters of two irreps. (The first column gives the value on $\{1\}$, and the remaining columns the values on the other six conjugacy classes.) Determine the rest of the character table, including the orders of the group and its conjugacy classes.