Problem Set 11

Some of the problems use material we'll cover on Friday. R is a commutative ring (with 1, as always in this course).

(1) Find the nilradical of $\mathbb{Z}_n$ ($n \in \mathbb{N}$).

(2) If every maximal ideal in R is of the form (c), for some $c \in R$ with with $c^2 = c$, then R is Noetherian. [Hint: show that every primary ideal is maximal; use Cohen's theorem.]

(3) Show that in $\mathbb{Z}[x, y]$ the ideals (x^i, y^j) are all primary ideals belonging to the prime ideal (x, y).

(4) Find a reduced primary decomposition for the ideal $I = (x^2, xy, 2)$ in $\mathbb{Z}[x, y]$ and determine the associated primes of the primary ideals appearing in this decomposition.

(5) A prime ideal $P \subset R$ is called a *minimal prime* ideal of the ideal I if $I \subset P$ and there is no prime ideal P' such that $I \subset P' \subsetneq P$. (a) If an ideal I of R is contained in a prime ideal P of R, show (using Zorn's lemma) that P contains a minimal prime ideal of I. (b) Show that every proper ideal has at least one minimal prime ideal. (c) Show that $Rad(I)$ is the intersection of all the minimal primes of I.

(6) If N is a P-primary submodule of an R-module M and $rx \in N$ ($r \in R$, $x \in M$), then either $r \in P$ or $x \in N$.

(7) If M is an R-module and $x \in M$, the annihilator of x, denoted $\text{ann}(x)$, is $\{r \in R \,|\, rx = 0\}$. (a) Show that $\text{ann}(x)$ is an ideal. (b) Assuming $M \neq 0$, show that a maximal element of the set $\{\text{ann}(x) \,|\, x \in M \backslash \{0\}\}$ of ideals is prime.

(8) Let R be Noetherian and $M \neq \{0\}$ an R-module. If P is prime, of the form $\text{ann}(x)$ for some $x \in M$, then P is called an *associated prime* of M. (a) Using (7)(b), show that an associated prime exists. (b) If M satisfies the ACC, prove that there exist primes $P_1, \ldots, P_{r-1}$ and a sequence of submodules $M = M_1 \supset M_2 \supset \cdots \supset M_r = \{0\}$ such that $M_i/M_{i+1} \cong R/P_i$ for each $i < r$.

(9) Continuing (8), show that the following conditions on $r \in R$ are equivalent: (i) for each $x \in M$ there exists $n(x) \in \mathbb{N}$ such that $r^{n(x)}x = 0$; (ii) r lies in every associated prime of M.